Matching Equivalences on Higher Dimensional Automata Models*

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Abstract

The intention of the paper is to show the applicability of the general categorical framework of open maps to the setting of two models – higher dimensional automata (HDA) and timed higher dimensional automata (THDA) – in order to transfer general concepts of equivalences to the models. First, we define categories of the models under consideration, whose morphisms are to be thought of as simulations. Then, accompanying (sub)categories of observations are chosen relative to which the corresponding notions of open maps are developed. Finally, we use the open maps framework to obtain two abstract bisimulations which are established to coincide with hereditary history preserving bisimulations on HDA and THDA, respectively.

1 Introduction

Geometrical methods in concurrency theory have appeared recently for modelling, analysis and verification of the behaviour of concurrent systems. The most popular geometric model for concurrency is higher dimensional automata (HDA) which have been proposed by V. Pratt [21]. Actually at about the same time a bisimulation semantics has been given for HDA in [6]. Based on the concepts of HDA, numerous papers have emerged in the literature. Basic strands of research are concerned with giving true concurrent semantics to concurrent languages [11, 8, 2], with analyzing correctness of distributed databases [3], with formalizing the fault-tolerant implementation of distributed programs [12, 10, 13]. The relationships between higher dimensional automata and other true concurrent models have been thoroughly studied in the paper [7]. Real-time extensions of HDA (THDA) have been investigated by Goubault [9].

In an attempt to explain and unify apparent differences between the extensive amount of research within the field of bisimulation equivalences, several category theoretic approaches to the matter have appeared. One of them was initiated by Joyal, Nielsen, and Winskel in [15] where they proposed an abstract

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way of capturing the notion of bisimulation through the so-called spans of open maps: first, a category of models of computations is chosen, then a subcategory of observation is chosen relative to which open maps are defined; two models are bisimilar if there exists a span of open maps between the models. The abstract definition of bisimilarity makes possible a uniform definition of bisimulation over different models ranging from interleaving models like transition systems [18] to true concurrency models like event structures [15], Petri nets [19], transition systems with independence [15], higher dimensional transition systems [23], higher dimensional automata [4]. The papers [14], and [25] transfer the concepts of abstract bisimularity to timed models — timed transition systems and timed event structures, respectively.

The contribution of the paper is to show the applicability of the general categorical framework of open maps to provide abstract characterizations of hereditary history preserving bisimulations in the setting of two models – HDA and THDA. In addition to the possibility of a uniform definition of bisimulation over different models presented as categories, the open maps based bisimilarity allows one to apply general results from the categorical setting (e.g. the existence of canonical models and characteristic games and logics) to concrete behavioural equivalences. In contrast to [4], we treat the notion of hereditary history preserving bisimulation [1, 7] but not bisimulation [17].

The rest of the paper is organized as follows. The following two sections concentrate on HDA and THDA, respectively. In particular, we, first, introduce categories of the models and, then, relate them. Further, we provide subcategories of observations of the categories to which the corresponding notions of open maps are developed. After that, we give a behavioural characterizations to the notion of open maps. Finally, the abstract equivalences based on spans of the open maps are shown to coincide with hereditary history preserving bisimulations on HDA and on THDA, respectively. Section 4 contains conclusion and some remarks on future work. This paper is a full version of [20].

2 (Untimed) HDA

2.1 The category HDA

In this section, we present the model of higher dimensional automata (HDA) – a geometric model for true concurrency based on the ideas of the works by V. Pratt [21] and R. van Glabbeek [6]. HDA are generalizations of the usual models of automata, also known as process graphs, state transition diagrams or labelled transition systems. The basic idea of HDA is to use the higher dimensions to represent the concurrent execution of processes. In contrast to interleaving models, HDA are built as sets of 0-cubes (points) and 1-cubes (edges) but also as sets of 2-cubes (squares), 3-cubes (cubes) and more generally n-cubes (hypercubes). In this way, an n-cube represents concurrent executions of n actions, whereas the edges of this cube depict the mutually exclusive execution of these n actions. For example, for two actions a and b, we model their concurrent execution by the square x labelled by $\{a, b\}$ and delineated by the 1-cubes x_1, y_1 and x_2, y_2 (in some sense, x_2 and y_2 are copies of x_1 and y_1, respectively), as shown on the right side of Figure 1. On the other hand, a mutually exclusive execution of a and b is modelled by the HDA generated by the 1-cubes x_1, y_1 and x_2, y_2 as shown on the left side of Figure 1. Thus, in HDA non-determinism arises as holes but concurrency is modelled by hypercubes with the interior filled. It is natural to graphically represent n-cubes as n-dimensional objects whose boundaries are the $(n-1)$-cubes from which the n-cubes can start and to which they end up. The 2-cube x shown on the right side of Figure 1 can start from x_1 or y_1. Similarly, x ends up to x_2 and y_2. Thus, the boundary of the square can be divided into two source boundary functions d_1^0 with $d_1^0(x) = x_1$ and d_2^0 with $d_2^0(x) = y_1$, and two target boundary functions d_1^1 with $d_1^1(x) = x_2$ and d_2^1 with $d_2^1(x) = y_2$. In addition, we fix a distinguished basepoint called the *initial point* and denoted as i_0.

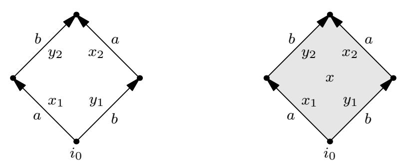

Figure 1: An example of concurrent and mutually exclusive executions of actions a and b in an HDA.

The following is the (well known but presented in a slightly different manner) definition of HDA from [7].

Definition 1. A *precubical set* M is a collection of pairwise disjoint sets $(M_n)_{n \in \mathbb{N}}$ together with boundary mappings $M_{n+1} \overset{d_i^0}{\underset{d_j^1}{\rightrightarrows}} M_n$ $(i, j = 1 \ldots (n+1))$ satisfying the commutativity of diagrams

$$\begin{array}{ccc} M_{n+2} & \xrightarrow{d_j^m} & M_{n+1} \\ \Big\downarrow{\scriptstyle d_i^k} & & \Big\downarrow{\scriptstyle d_i^k} \\ M_{n+1} & \xrightarrow[d_{j-1}^m]{} & M_n \end{array}$$

for all $i < j$ and $k, m = 0, 1$.

Definition 2. A *(labelled non-degenerate) HDA* is a triple $\mathrm{M} = (M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})$, where

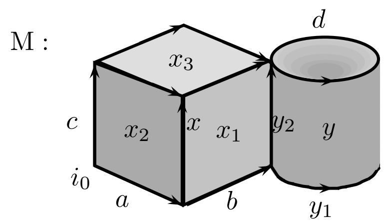

Figure 2: An example of an HDA M.

- M is a precubical set possessing the non-degeneracy property: for all $x \in M_{n+1}$ and $m = 0, 1$ it holds $|\{d_i^m(x) \mid i = 1 \dots n\}| = n$,
- $i_0^{\mathrm{M}} \in M_0$ is a distinguished basepoint of M, called the *initial point*,
- $l_L^{\mathrm{M}} : M_1 \to L$ is a *labelling function* from the 1-cubes of M to a set L of actions such that $l_L^{\mathrm{M}}(d_i^0(x)) = l_L^{\mathrm{M}}(d_i^1(x))$ for all $i = 1, 2$ and $x \in M_2$.

Whenever no confusion is possible we drop subscripts and superscripts on $\mathrm{M} = (M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})$ and write $\mathrm{M} = (M, i_0, l)$ instead, to denote an HDA M over a set L of actions.

Remark 1. Assume $\mathrm{M} = (M, i_0, l)$ to be an HDA over a set L of actions. For an n-cube x with $n > 1$, the 1-cubes $d_1^{\varepsilon_1^i} \circ \dots \circ d_{i-1}^{\varepsilon_{i-1}^i} \circ d_{i+1}^{\varepsilon_{i+1}^i} \circ \dots \circ d_n^{\varepsilon_n^i}(x)$, with $\varepsilon_j^i = 0, 1$, $1 \le j \le n$ and $j \ne i$, represent the same action $l_i(x) = l(d_1^{\varepsilon_1^i} \circ \dots \circ d_{i-1}^{\varepsilon_{i-1}^i} \circ d_{i+1}^{\varepsilon_{i+1}^i} \circ \dots \circ d_n^{\varepsilon_n^i}(x))$, since $l_L^{\mathrm{M}}(d_r^0(y)) = l_L^{\mathrm{M}}(d_r^1(y))$ for all $r = 1, 2$ and $y \in M_2$. So, we can extend the labelling function to all cubes in M by taking for $x \in M_n$ an action $l(x) = (l_1(x), \dots, l_n(x))$, if $n > 1$, and $l(x) = \emptyset$, if $n = 0$.

Example 1. To illustrate the concept specified in Definition 2, consider the HDA $\mathrm{M} = (M, i_0, l)$ over $L = \{a, b, c, d\}$, depicted in Figure 2. M contains the 3-cube x and the 2-cube y convoluted to the cylinder. To define the boundaries of x and y we put $x_1 = d_1^1(x)$, $x_2 = d_2^0(x)$, $x_3 = d_3^1(x)$, $y_1 = d_1^0(y)$ and $y_2 = d_2^0(y)$. Clearly, M possesses the non-degeneracy property. The initial point is $i_0 \in M_0$. The actions of the edges of x and y are given by $l(d_2^0(d_3^0(x))) = a$, $l(d_1^0(d_3^0(x))) = b$, $l(d_1^0(d_2^0(x))) = c$ and $l(d_1^0(y)) = d$.

Define a morphism between two HDA mapping cubes and actions of the simulated system to simulating cubes and actions of the other and satisfying some requirements.

Definition 3. Let $\mathrm{M} = (M, i_0^{\mathrm{M}}, l_{L^{\mathrm{M}}}^{\mathrm{M}})$ and $\mathrm{N} = (N, i_0^{\mathrm{N}}, l_{L^{\mathrm{N}}}^{\mathrm{N}})$ be HDA. A mapping $\mathrm{f} = \langle f, \alpha \rangle$ (where $f = \cup f_n$, $f_n : M_n \to N_n$ and $\alpha : L^{\mathrm{M}} \to L^{\mathrm{N}}$) is called a *morphism* from M to N iff it holds:

1. $f_0(i_0^{\mathrm{M}}) = i_0^{\mathrm{N}}$,

2. $l^{\mathrm{N}}_{L^{\mathrm{N}}} \circ f = \alpha \circ l^{\mathrm{M}}_{L^{\mathrm{M}}}$,

3. $f_n \circ d_i^m = d_i^m \circ f_{n+1}$.

The first condition guarantees that morphisms preserve initial points; the second and third conditions ensure the consistency of actions and boundaries of cubes, respectively.

HDA with morphisms between them form a category **HDA** in which the composition of two morphisms $\mathrm{f} = \langle f, \alpha \rangle : \mathrm{M} \to \mathrm{M}'$ and $\mathrm{g} = \langle g, \beta \rangle : \mathrm{M}' \to \mathrm{M}''$ is $\mathrm{g} \circ \mathrm{f} = \langle g \circ f,\, \beta \circ \alpha \rangle : \mathrm{M} \to \mathrm{M}''$, and the identity morphism is a pair of the identity mappings.

2.2 Hereditary history preserving bisimulation

In order to reason about the behaviour of HDA, we introduce the following notions and notations. A *cubical path* in an HDA M is a sequence $P = p_0 p_1 \dots p_k$[1] of cubes such that $p_{s-1} = d_i^0(p_s)$ or $p_s = d_j^1(p_{s-1})$ for all $p_s \in M$, $s = 1 \dots k$, and, moreover, $p_0 = i_0^{\mathrm{M}}$. A cubical path $P = p_0 p_1 \dots p_k$ is *acyclic* if there are no other relations between the p_r and $p_{r'}$ ($0 \le r < r' \le k$) than the relations above. For cubical paths $P = p_0 \dots p_k$ and $Q = q_0 \dots q_n$, we say that Q is an extension of P (denote $P \to Q$) if $n \ge k$ and $p_0 \dots p_k = q_0 \dots q_k$. In particular, we write $P \xrightarrow{d_i^m} Q$ if $n = k+1$ and either $q_k = d_i^0(q_{k+1})$ for $m = 0$ or $q_{k+1} = d_i^1(q_k)$ for $m = 1$. Further, $\mathcal{CP}(\mathrm{M})$ ($\mathcal{CP}_u(\mathrm{M})$) is the set of all cubical paths (ending with a cube u) in M. An n-cube x in M is called *reachable* if there exists some $P \in \mathcal{CP}_x(\mathrm{M})$. For a cubical path $P = p_0 \dots p_k$ in an HDA $\mathrm{M} = (M, i_0, l_L)$, define the structure $\mathrm{M}' = (M', i_0, l_L|_{(M')_1})$ with $(M')_n = \{d_{i_1}^{\alpha_1} \circ \dots \circ d_{i_l}^{\alpha_l}(p_i) \mid \alpha_j = 0, 1, 1 \le j \le l, 1 \le i_1 < \dots < i_l \le \dim p_i, 1 \le l \le \dim p_i, 1 \le i \le k\} \cup \{p_i \mid 0 \le i \le k\} \subseteq M_n$. It is easy to verify that M' is an HDA, and, moreover, a sub-HDA of M. In this case, M' is said to *have the form of the cubical path* P in the HDA M.

We proceed with some kind of equivalence on cubical paths [7]. A *homotopy* (denote $\sim$) is the least equivalence on cubical paths in M such that if P and P' are *s-adjacent* (denote $P \overset{s}{\leftrightarrow} P'$), i.e. P' can be obtained from P by replacing (for $i < j$ and $m = 0, 1$)

$$\text{either a segment } \xrightarrow{d_i^0} p_s \xrightarrow{d_j^m} \text{ by a segment } \xrightarrow{d_{j-1}^m} p'_s \xrightarrow{d_i^0}, \text{ or vice versa;}$$

$$\text{or a segment } \xrightarrow{d_j^m} p_s \xrightarrow{d_i^1} \text{ by a segment } \xrightarrow{d_i^1} p'_s \xrightarrow{d_{j-1}^m}, \text{ or vice versa,}$$

then P and P' are equivalent. Moreover, P and P' are (s, u, v)-*adjacent* (denote $P \overset{(s,u,v)}{\longleftrightarrow} P'$), if P' can be obtained from $P = \hat{p}_0 \dots \hat{p}_s \dots \hat{p}_k$ by an s-adjacency replacement of the segment $\xrightarrow{d_u^n} \hat{p}_s \xrightarrow{d_v^l}$. For every $P \in \mathcal{CP}(\mathrm{M})$ we write $[P]$ to denote its homotopy class.

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[1] In case we need a detailed presentation of P we shall write $P = p_0 \xrightarrow{d_{j_1}^{m_1}} \dots \xrightarrow{d_{j_k}^{m_k}} p_k$, where $d_{j_i}^{m_i}(p_i) = p_{i-1}$ if $m_i = 0$ and $d_{j_i}^{m_i}(p_{i-1}) = p_i$ if $m_i = 1$, for all $1 \le i \le k$.

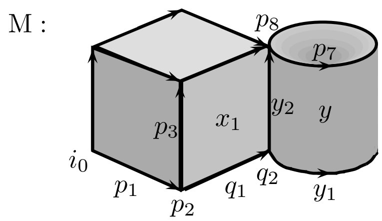


Figure 3: Cubical paths in the HDA M.

Example 2. Recall the HDA M from Example 1. The sequences $P = i_0p_1p_2p_3x_1 y_2yp_7p_8p_7$ and $Q = i_0p_1p_2q_1q_2y_2yp_7p_8p_7$, shown in Figure 3, are cubical paths in M. P and Q are homotopic cubical paths since $P \overset{4}{\leftrightarrow} (i_0p_1p_2q_1x_1y_2yp_7p_8p_7) \overset{5}{\leftrightarrow} Q$. An example of an acyclic cubical path is the sequence $i_0p_1p_2p_3x_1y_2$.

We proceed by considering the following fact which is a slight modification of Proposition 2 from [7].

Lemma 1. *Given a segment* $\xrightarrow{d_u^0} p_s \xrightarrow{d_v^1}$ *with* $u \neq v$*, or a segment* $\xrightarrow{d_u^0} p_s \xrightarrow{d_v^0}$ *in a cubical path* $P = p_0 \dots p_k$ *in an HDA* M*, there is a unique path* P' *in* M *such that* $P \overset{s}{\longleftrightarrow} P'$*.*

Remark 2. Intuitively, in $P = p_0 \dots p_k \in \mathcal{CP}(\mathrm{M})$ every segment $p_{s-1} \xrightarrow{d_u^\lambda} p_s$ represents either the start of the action $l_u(p_s)$, if $\lambda = 0$, or the termination of the action $l_u(p_{s-1})$, if $\lambda = 1$. Having the start of the action a in P, i.e. $p_{r-1} \xrightarrow{d_{u_r}^0} p_r$ with $l_{u_r}(p_r) = a$, we are going to find the termination of the action in P, i.e. $p_{t-1} \xrightarrow{d_{v_t}^1} p_t$ with $l_{v_t}(p_{t-1}) = a$, if any. Suppose $\xrightarrow{d_u^0} q_s \xrightarrow{d_v^\mu}$ in an arbitrary cubical path Q in M. Two cases are admissible. First, let $\mu = 0$. Then, there exists a unique cubical path Q' in M such that $Q \overset{s}{\longleftrightarrow} Q'$, due to Lemma 1. Next, let $\mu = 1$. If $u \neq v$, the case is similar to that when $\mu = 0$. If $u = v$, the action $l_u(q_s)$ starts, and then, the same occurrence of $l_u(q_s)$ terminates. This means that for all cubical path Q' in M it holds that $Q \overset{s}{\not\longleftrightarrow} Q'$. By the repeated applications of the above facts, we can construct a unique adjacency-chain of the form: either $P \overset{r}{\longleftrightarrow} P_{r+1} \overset{r+1}{\longleftrightarrow} \dots \overset{t-2}{\longleftrightarrow} P_{t-1} \overset{t-1}{\not\longleftrightarrow} P_t$, if the termination $p_{t-1} \xrightarrow{d_{v_t}^1} p_t$ of a is in P, or $P \overset{r}{\longleftrightarrow} P_{r+1} \overset{r+1}{\longleftrightarrow} \dots \overset{k-2}{\longleftrightarrow} P_{k-1} \overset{k-1}{\longleftrightarrow} P_k$, if there is no termination of a in P.

Now, we need to introduce some auxiliary notions and notations. For a cubical path $P \in \mathcal{CP}_{p_k}(\mathrm{M})$ with $\dim p_k > 0$, define its i-beginning $d_i^0(P)$ to be a cubical path from $\mathcal{CP}_{d_i^0(p_k)}(\mathrm{M})$ such that either (i) $P = d_i^0(P)p_k$ or (ii) $P \overset{m+1}{\longleftrightarrow} P_1 \overset{m+2}{\longleftrightarrow} \dots \overset{k-2}{\longleftrightarrow} P_{k-m-2} \overset{k-1}{\longleftrightarrow} d_i^0(P)p_k$ for some $0 \leq m \leq k-2$. Also, define the i-ending $d_i^1(P)$ of P to be a cubical path $d_i^1(P) \in \mathcal{CP}_{d_i^1(p_k)}(\mathrm{M})$ such that $d_i^1(P) = Pd_i^1(p_k)$.

Lemma 2. *Given an HDA* M *and a cubical path* $P \in \mathcal{CP}_{p_k}(\mathrm{M})$ *with* $\dim p_k > 0$*, there exists a unique cubical path* $d_i^l(P) \in \mathcal{CP}_{d_i^l(p_k)}(\mathrm{M})$ *(*$l = 0, 1$*).*

Proof. W.l.o.g. assume $l = 0$. Clearly, cases (i) and (ii) of the definition of i-beginning can not be fulfilled simultaneously. Consider the proof when case (ii) holds (the proof when case (i) holds is obvious). Contemplate $P = p_0 \ldots p_k \in \mathcal{CP}_{p_k}(\mathrm{M})$ with $\dim p_k = n > 0$. It may happen that different occurrences of an action can appear in P (for example, an auto-concurrent action). We distinguish the different occurrences by indexing them. Hence, we can assume that there is at most one occurrence of an action in P.

Consider the cube p_k. It represents a simultaneous execution of n actions $l_1(p_k), \ldots, l_n(p_k)$. Then, due to the definition of a cubical path, there exists a unique number $m = m(P, i)$ such that the segment $p_m \xrightarrow{d^0_{r_{m+1}}} p_{m+1}$ in P represents the start of the action $l_{r_{m+1}}(p_{m+1}) = l_i(p_k)$. Since P ends with p_k, there is no termination of the action $l_i(p_k)$ in P. By Remark 2, we can construct a unique adjacency-chain $(P = P_{m+1}) \overset{m+1}{\longleftrightarrow} \cdots \overset{k-1}{\longleftrightarrow} P_k$ in M. Clearly, if $Q \overset{t}{\longleftrightarrow} Q'$, i.e. a segment $\xrightarrow{d^0_v} q_t \xrightarrow{d^\varepsilon_w}$ is replaced by a segment $\xrightarrow{d^\varepsilon_{w'}} q'_t \xrightarrow{d^0_{v'}}$, then $l_v(q_t) = l_{v'}(q_{t+1})$ in M. Using this fact for every $P_s \overset{s}{\longleftrightarrow} P_{s+1}$ with $(m+1) \le s \le (k-1)$, we get that $l_{r_{m+1}}(p_{m+1}) = l_{r^k_k}(p_k)$, where the cubical path P_k ends with $p'_{k-1} \xrightarrow{d^0_{r^k_k}} p_k$. Having the coincidence of the actions $l_i(p_k)$, $l_{r_{m+1}}(p_{m+1})$ and $l_{r^k_k}(p_k)$, we conclude that $i = r^k_k$, due to M possessing the non-degeneracy property. Hence, $d_i^0(P)$ defined by $P_k = d_i^0(P)p_k$, is a cubical path in M satisfying the considered condition of the definition of i-beginning and, moreover, $d_i^0(P)$ is unique. □

Example 3. To illustrate the concepts of i-beginning and i-ending of a cubical path P, consider the HDA M from the Example 1. Contemplate $P = i_0 \xrightarrow{d^0_1} p_1 \xrightarrow{d^1_1} p_2 \xrightarrow{d^0_1} p_3 \xrightarrow{d^0_1} x_1 \xrightarrow{d^1_1} y_2 \xrightarrow{d^0_2} y \in \mathcal{CP}(\mathrm{M})$ shown in Figure 3. Since $\dim y = 2 > 0$, we can find i-beginning of P for any $1 \le i \le \dim y = 2$. According to the definition of i-beginning, it is required to be from $\mathcal{CP}_{d^0_i(y)}(\mathrm{M})$. We start with $i = 1$. One can see that the cube $d^0_1(y) = y_1$ doesn't belong to P and, hence, case (ii) holds in the definition of i-beginning. Find the number $m = m(P, 1)$ using the reasonings in the proof of Lemma 2. Consider the action $l_1(y) = c$. There exists a unique segment $p_2 \xrightarrow{d^0_1} p_3$ in P such that it represents the start of the action $l(p_3) = c = l_1(y)$ in P. Hence, $m = 2$. Then, we have the adjacency-chain $P \overset{3}{\longleftrightarrow} P_1 \overset{4}{\longleftrightarrow} P_2 \overset{5}{\longleftrightarrow} d^0_1(P)y$ in M. It is easy to see that $P_1 = i_0p_1p_2q_1x_1y_2y$ and $P_2 = i_0p_1p_2q_1q_2y_2y$. So, $d^0_1(P) = i_0p_1p_2q_1q_2y_1$. We proceed with $i = 2$. Obviously, the cube $d^0_2(y) = y_2$ belongs to P. Hence, to define its 2-beginning we have to use case (i) in the definition of i-beginning. Then, $d^0_2(P) = i_0p_1p_2p_3x_1y_2$. Clearly, the 1-ending and 2-ending of P are $d^0_1(P) = i_0p_1p_2p_3x_1y_2yp_7$ and $d^0_1(P) = i_0p_1p_2p_3x_1y_2yy_2$, respectively.

The following fact clarifies why the morphisms between HDA are simulations.

Lemma 3. *Given a morphism* $\mathrm{f} = \langle f, \alpha \rangle : \mathrm{M} \to \mathrm{N}$ *in* **HDA**, *for all* $P = p_0 \xrightarrow{d_{i_1}^{\epsilon_1}} \ldots \xrightarrow{d_{i_k}^{\epsilon_k}} p_k \in \mathcal{CP}(\mathrm{M})$ *it holds:*

1. *there exists a unique* $f(P) = f(p_0) \xrightarrow{d_{i_1}^{\epsilon_1}} \ldots \xrightarrow{d_{i_k}^{\epsilon_k}} f(p_k) \in \mathcal{CP}(\mathrm{N})$;
2. *whenever* $P \xrightarrow{d_i^l} P'$ *in* M, *then* $f(P) \xrightarrow{d_i^l} f(P')$ *in* N;
3. *whenever* $P \overset{(s,u,v)}{\longleftrightarrow} P'$ *in* M, *then* $f(P) \overset{(s,u,v)}{\longleftrightarrow} f(P')$ *in* N.

Proof. Obvious. □

Further, we define a behavioural equivalence on HDA, called hereditary history preserving bisimulation (hhp-bisimulation), which is in close similarity with the corresponding definition from [7].

Definition 4. Let M and N be HDA.

Cubical paths $P = p_0 \ldots p_k$ in M and $Q = q_0 \ldots q_k$ in N are called *l-related* iff $l^{\mathrm{M}}(p_j) = l^{\mathrm{N}}(q_j)$ for all $j = 0 \ldots k$.

A binary relation $\mathcal{R}$ on cubical paths in M and N is called a *hereditary history preserving bisimulation (hhp-bisimulation)* between M and N if for any $(P, Q) \in \mathcal{R}$, P and Q are l-related and the following conditions are satisfied:

1. if $P \xrightarrow{d_i^m} P'$ then $Q \xrightarrow{d_i^m} Q'$ and $(P', Q') \in \mathcal{R}$ for some Q' in N,
2. if $Q \xrightarrow{d_i^m} Q'$ then $P \xrightarrow{d_i^m} P'$ and $(P', Q') \in \mathcal{R}$ for some P' in M,
3. if $P' \to P$ then $Q' \to Q$ and $(P', Q') \in \mathcal{R}$ for some Q' in N,
4. if $Q' \to Q$ then $P' \to P$ and $(P', Q') \in \mathcal{R}$ for some P' in M,
5. if $P \overset{(s,u,v)}{\longleftrightarrow} P'$ then $Q \overset{(s,u,v)}{\longleftrightarrow} Q'$ and $(P', Q') \in \mathcal{R}$ for some Q' in N,
6. if $Q \overset{(s,u,v)}{\longleftrightarrow} Q'$ then $P \overset{(s,u,v)}{\longleftrightarrow} P'$ and $(P', Q') \in \mathcal{R}$ for some P' in M.

HDA M and N are *hhp-bisimilar* if there exists an hhp-bisimulation between them which relates their initial points (regarded as cubical paths).

Note, hhp-bisimulation is indeed an equivalence relation.

Example 4. To get more intuition about the above concept, we consider examples of hhp-bisimular and non-hhp-bisimular HDA. First, contemplate the HDA shown in Figure 4. The boundary functions are given as follows: $d_1^0(x_1) = p_1$, $d_2^1(x_1) = p_3$, $d_1^0(x_2) = p_2$, $d_2^1(x_2) = p_4$ in the left-hand HDA, and $d_1^0(y) = q_1$, $d_2^1(y) = q_2$ in the right-hand HDA. Take cubical paths $P_1 = sp_1s_1p_3s_3p_5s_5p_7s_7$ and $P_2 = sp_2s_2p_4s_4p_6s_6p_8s_8$ in the left-hand HDA and cubical paths $Q_1 = rq_1r_1q_2r_2q_3r_3q_4r_4$ and $Q_2 = rq_1r_1q_2r_2q_5r_5q_6r_4$ in the right-hand HDA. It is easy to see that these HDA are hhp-bisimilar because a required hhp-bisimulation $\mathcal{R}$

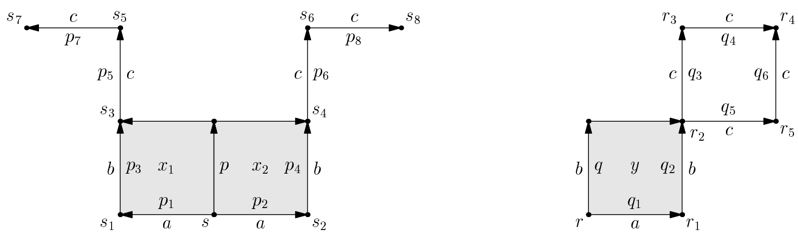


Figure 4: An example of hhp-bisimular HDA.

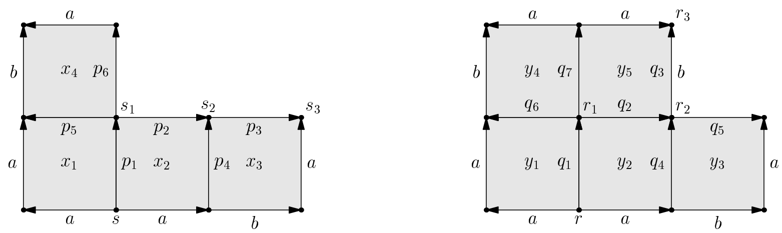


Figure 5: An example of non-hhp-bisimular HDA.

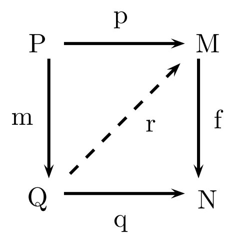
can be constructed from the set $\{(P_1, Q_1), (P_2, Q_2), (P_1, Q_2), (P_2, Q_1)\}$ using conditions 1-6 of Definition 4. Next, consider the HDA shown in Figure 5. The boundary functions are given as follows: $d_1^0(x_1) = p_1$, $d_2^1(x_1) = p_5$, $d_1^0(x_2) = p_1$, $d_2^1(x_2) = p_2$, $d_2^0(x_3) = p_4$, $d_1^1(x_3) = p_3$, $d_1^0(x_4) = p_5$, $d_2^0(x_4) = p_6$ in the left-hand HDA, and $d_1^0(y_1) = q_1$, $d_2^1(y_1) = q_6$, $d_1^0(y_2) = q_1$, $d_2^1(y_2) = q_2$, $d_2^0(y_3) = q_4$, $d_1^1(y_3) = q_5$, $d_1^0(y_4) = q_6$, $d_2^0(y_4) = q_7$, $d_2^0(y_5) = q_2$, $d_1^1(y_5) = q_3$ in the right-hand HDA. We then have that the cubical path $(sp_1s_1p_2s_2p_3s_3)$ in the left-hand HDA could be related only to the cubical path $(rq_1r_1q_2r_2q_3r_3)$ in the right-hand HDA. Moreover, we can see that $(rq_1r_1q_2r_2q_3r_3) \overset{(5,1,1)}{\longleftrightarrow} (rq_1r_1q_2y_5q_3r_3)$ in the right-hand HDA. Then, there should exist a cubical path P in the left-hand HDA such that $(sp_1s_1p_2s_2p_3s_3) \overset{(5,1,1)}{\longleftrightarrow} P$ but it is not the case.

2.3 Open Maps Characterization

In this subsection, we develop a notion of open morphism in the category **HDA**, give an alternative characterization of openness and establish the coincidence between abstract bisimulation (based on open morphisms) and hhp-bisimulation on HDA.

Consider the notion of open maps from [15]. Let **M** be a category of models and **P** be a subcategory of observation.

Definition 5. A morphism f : M → N in **M** is called **P**-*open*, if for any morphism m : P → Q in **P** and any commutative square in **M** depicted below



there exists a morphism $\mathrm{r} : \mathrm{Q} \to \mathrm{M}$ splitting the diagram on the two commutative triangles.

It is easy to verify that the identity morphism and the composition of **P**-open morphisms in **M** are **P**-open morphisms in **M**. So, objects in the category **M** and **P**-open morphisms can form a subcategory of the category **M**.

As reported in [15], the open map approach provides general concepts of bisimilarity for any categorical model of computation. The definition is given in terms of a spans of open maps. Two models M' and M'' in **M** are said to be **P**-*bisimilar* if there exists a span $\mathrm{M}' \xleftarrow{\mathrm{f}'} \mathrm{M} \xrightarrow{\mathrm{f}''} \mathrm{M}''$ with vertex M and **P**-open morphisms f', f''.

We consider **HDA** as a category of models. Relying on the standards of HDA, we choose an observation of an HDA M to be an HDA $\mathrm{M_P}$ having the form of an acyclic cubical path P in the HDA M. We use **cP** to denote the full subcategory of observations of the category **HDA**.

For our purposes we need to endow **HDA** with a fibred structure. Denote $\mathbf{HDA}_L$ the subcategory of **HDA** whose objects are HDA labelled over L and morphisms have the identity action component. We shall follow similar conventions for the other categories defined in the paper.

We next associate every cubical path in an HDA with a morphism whose domain is an observation and codomain is the HDA.

Lemma 4. *Let* M *be an object in* $\mathbf{HDA}_L$, P *be a cubical path in* M *and* $\mathrm{M_P}$ *be a sub-HDA of* M *having the form of* P. *Then, there exists a morphism* $\mathrm{p} = \langle p, 1_L \rangle : \mathrm{M}_{\widetilde{\mathrm{P}}} \to \mathrm{M_P} \hookrightarrow \mathrm{M}$ *in* $\mathbf{HDA}_L$, *where* $\mathrm{M}_{\widetilde{\mathrm{P}}}$ *is an observation.*

Proof. W.l.o.g. assume that $P = p_0 \dots p_k$ and $\mathrm{M_P} = (M_P, i_0, l_L)$. Set $A = \{O \in \mathcal{CP}(\mathrm{M_P}) \mid \exists \widehat{O} \in \mathcal{CP}(\mathrm{M_P}) \text{ s. t. } O \to \widehat{O} \text{ and } [\widehat{O}]_{\mathrm{M_P}} = [P d^1_{\dim p_k}(p_k) \dots d^1_1 \circ \dots \circ d^1_{\dim p_k}(p_k)]_{\mathrm{M_P}}\}$. Here, for a cubical path $O' \in \mathcal{CP}(\mathrm{M_P})$, $[O']_{\mathrm{M_P}}$ denotes its homotopic class containing cubical paths from $\mathcal{CP}(\mathrm{M_P})$. Define a structure $\mathrm{M}_{\widetilde{\mathrm{P}}} = \{M_{\widetilde{P}}, \tilde{i}_0, \tilde{l}_L\}$ with

- $(M_{\widetilde{P}})_n = \{[O = o_0 \dots o_r]_{\mathrm{M_P}} \mid O \in A \text{ and } o_r \in (M_P)_n\}$ with $\widetilde{d}^l_i([O]_{\mathrm{M_P}}) = [d^l_i(O)]_{\mathrm{M_P}}$ for $[O]_{\mathrm{M_P}} \in (M_{\widetilde{P}})_n$ and $n > 0$,
- $\tilde{i}_0 = [i_0]_{\mathrm{M_P}}$,
- $\tilde{l}_L([o_0 \dots o_r]_{\mathrm{M_P}}) = l_L(o_r)$ for all $[o_0 \dots o_r]_{\mathrm{M_P}} \in (M_{\widetilde{P}})_1$.

We shall prove that $\mathrm{M}_{\widetilde{\mathrm{P}}}$ is indeed an HDA. First, consider an arbitrary $[O = o_0 \dots o_r]_{\mathrm{M_P}} \in (M_{\widetilde{P}})_n$ $(n > 0)$ and show that $\widetilde{d}_i^l([O]_{\mathrm{M_P}}) \in (M_{\widetilde{P}})_{n-1}$. According to the definition of $\mathrm{M}_{\widetilde{\mathrm{P}}}$, $O \in A$, i.e. $O \in \mathcal{CP}(\mathrm{M_P})$ and there exists $\widehat{O}$ such that $O \to \widehat{O}$ and $[\widehat{O}]_{\mathrm{M_P}} = [Pd^1_{\dim p_k}(p_k) \dots d^1_1 \circ \dots \circ d^1_{\dim p_k}(p_k)]_{\mathrm{M_P}}$, and, moreover, $o_r \in (M_P)_n$. W.l.o.g. assume $l = 0$. Since $\mathrm{M_P}$ is an HDA, $O \in \mathcal{CP}(\mathrm{M_P})$ implies $d^0_i(O) \in \mathcal{CP}(\mathrm{M_P})$ due to Lemma 2. Let $d^0_i(O) = o_0 \dots o_m o'_{m+1} \dots o'_{r-2} d^0_i(o_r)$. Also, let $\widehat{O} = o_0 \dots o_r o_{r+1} \dots o_{k+\dim p_k}$. Then, there exists $\check{O} = o_0 \dots o_m o'_{m+1} \dots o'_{r-2} d^0_i(o_r) o_r o_{r+1} \dots o_{k+\dim p_k} \in \mathcal{CP}(\mathrm{M_P})$ such that $d^0_i(O) \to \check{O}$ and $[\check{O}]_{\mathrm{M_P}} = [\widehat{O}]_{\mathrm{M_P}} = [Pd^1_{\dim p_k}(p_k) \dots d^1_1 \circ \dots \circ d^1_{\dim p_k}(p_k)]_{\mathrm{M_P}}$. Obviously, $d^0_i(o_r) \in (M_P)_{n-1}$. Thus, $\widetilde{d}_i^l([O]_{\mathrm{M_P}}) \in (M_{\widetilde{P}})_{n-1}$. We proceed with showing that the diagrams in Definition 1 commute, i.e. $\widetilde{d}_i^\alpha(\widetilde{d}_j^\beta([O])) = \widetilde{d}_{j-1}^\beta(\widetilde{d}_i^\alpha([O]))$, if $1 \le i < j \le n$, for all $[O] \in (M_{\widetilde{P}})_n$ with $n \ge 2$. Check the case $\alpha = 0$ and $\beta = 1$ (checking of the remaining cases is similar). Take an arbitrary $[O] \in (M_{\widetilde{P}})_n$ with $n \ge 2$. Let $O \in \mathcal{CP}_{o_r}(\mathrm{M_P})$. W.l.o.g. assume that $d^0_i(O)$ is obtained due to the fulfillment of case (ii) in the definition of i-beginning. Then, there exists the corresponding adjacency-chain $O \overset{m+1}{\longleftrightarrow} \dots \overset{r-1}{\longleftrightarrow} d^0_i(O) o_r$. We can extend every cubical path of the adjacency-chain with $d^1_j(o_r)$. This implies that we get a new adjacency-chain. Prolong it with r-adjacency to obtain the adjacency-chain $Od^1_j(o_r) \overset{m+1}{\longleftrightarrow} \dots \overset{r-1}{\longleftrightarrow} d^0_i(O) o_r d^1_j(o_r) \overset{r}{\longleftrightarrow} d^0_i(O) d^1_{j-1}(d^0_i(o_r)) d^1_j(o_r) = d^1_{j-1}(d^0_i(O)) d^1_j(o_r)$. On the other hand, we have $Od^1_j(o_r) = d^1_j(O)$. We know that i-beginning of $d^1_j(O)$ is required to be in $\mathcal{CP}_{d^0_i(d^1_j(o_r))}(\mathrm{M_P})$. Since the cube $d^0_i(d^1_j(o_r))$ doesn't belong to $d^1_j(O)$, for its i-beginning case (ii) holds. As $d^1_j(O)$ is an extension of O, the adjacency-chain, corresponding to $d^0_i(d^1_j(O))$, looks as $d^1_j(O) \overset{m+1}{\longleftrightarrow} \dots \overset{r}{\longleftrightarrow} d^0_i(d^1_j(O)) d^1_j(o_r)$. It coincides with the previous adjacency-chain, due to Lemma 1. Hence, $d^0_i(d^1_j(O)) = d^1_{j-1}(d^0_i(O))$. Thus, $\widetilde{d}^0_i(\widetilde{d}^1_j([O])) = \widetilde{d}^1_{j-1}(\widetilde{d}^0_i([O]))$. The non-degeneracy property of $\mathrm{M}_{\widetilde{\mathrm{P}}}$ immediately follows from the non-degeneracy property of M.

It is routine to show that $\mathrm{M}_{\widetilde{\mathrm{P}}}$ has the form of the cubical path $\widetilde{P} = [p_0][p_0 p_1] \dots [p_0 p_1 \dots p_k]$. Clearly, $\widetilde{P}$ is an acyclic cubical path. Hence, $\mathrm{M}_{\widetilde{\mathrm{P}}}$ is an observation. It remains to define a mapping $\mathrm{p} = \langle p, 1_L \rangle : \mathrm{M}_{\widetilde{\mathrm{P}}} \to \mathrm{M}$. Put $p([o_0 \dots o_r]) = o_r$ for all $[o_0 \dots o_r] \in (M_{\widetilde{P}})_n$ $(n \ge 0)$. Obviously, p is a morphism in $\mathbf{HDA}_L$. □

Our next aim is to characterize $\mathbf{cP}_L$-open morphisms in $\mathbf{HDA}_L$ relative to the subcategory $\mathbf{cP}_L$ defined prior to that. In the below characterization, the first condition is usually referred to as the "higher-dimensional" zig-zag property and the second one ensures that $\mathbf{cP}_L$-open morphisms reflect concurrency.

Theorem 1. *Given objects* M *and* M′ *in* $\mathbf{HDA}_L$*, a morphism* $\mathrm{f} = \langle f, 1_L \rangle : \mathrm{M} \to \mathrm{M}'$ *is* $\mathbf{cP}_L$*-open iff for all* $P \in \mathcal{CP}(\mathrm{M})$ *the following holds:*

1. *if* $f(P) \xrightarrow{d^l_i} Q'$ *in* M′*, then* $P \xrightarrow{d^l_i} P'$ *and* $f(P') = Q'$ *for some* $P' \in \mathcal{CP}(\mathrm{M})$*,*

2. *if* $f(P) \overset{(s,u,v)}{\longleftrightarrow} Q'$ *in* M'*, then* $P \overset{(s,u,v)}{\longleftrightarrow} P'$ *and* $f(P') = Q'$ *for some* $P' \in \mathcal{CP}(\mathrm{M})$.

Proof. ($\Rightarrow$) Assume $\mathrm{f} = \langle f, 1_L\rangle : \mathrm{M} \to \mathrm{M}'$ to be a $\mathbf{cP}_L$-open morphism. Consider the proof of item 1 (the proof of item 2 is similar). W.l.o.g. suppose that $P \in \mathcal{CP}(\mathrm{M})$ and $f(P) \xrightarrow{d_i^l} Q'$ in M'. Let $\mathrm{M_P}$ ($\mathrm{M_{Q'}}$) be a sub-HDA of M (M') having the form of P (Q'). By Lemma 4, there exists a morphism $\mathrm{p} = \langle p, 1_L\rangle : \mathrm{M}_{\widetilde{\mathrm{P}}} \to \mathrm{M}$ ($\mathrm{q} = \langle q, 1_L\rangle : \mathrm{M}_{\widetilde{\mathrm{Q}}'} \to \mathrm{M}'$) in $\mathbf{HDA}_L$ with an observation $\mathrm{M}_{\widetilde{\mathrm{P}}}$ ($\mathrm{M}_{\widetilde{\mathrm{Q}}'}$), specified in the Lemma. Notice, $p(\widetilde{P}) = P$ ($q(\widetilde{Q}') = Q'$).

W.l.o.g. assume that $\widetilde{P} = \tilde{p}_0 \ldots \tilde{p}_k$ and $\widetilde{Q}' = \tilde{q}_0 \ldots \tilde{q}_k \tilde{q}_{k+1}$. Set $m(\tilde{p}_j) = \tilde{q}_j$ and $m(d_{j_1}^{\alpha_1} \circ \cdots \circ d_{j_s}^{\alpha_s}(\tilde{p}_j)) = d_{j_1}^{\alpha_1} \circ \cdots \circ d_{j_s}^{\alpha_s}(\tilde{q}_j)$, for all $\alpha_r = 0, 1$, $1 \le r \le s$, $1 \le j_1 < \ldots < j_s \le \dim \tilde{p}_j$, $1 \le s \le \dim \tilde{p}_j$ and $0 \le j \le k$. It is easy to see that $\mathrm{m} = \langle m, 1_L\rangle : \mathrm{M}_{\widetilde{\mathrm{P}}} \to \mathrm{M}_{\widetilde{\mathrm{Q}}'}$ is a morphism in $\mathbf{cP}_L$. By the definition of m, we get $\mathrm{f} \circ \mathrm{p} = \mathrm{q} \circ \mathrm{m}$.

Due to f being a $\mathbf{cP}_L$-open morphism, there exists a morphism $\mathrm{r} : \mathrm{M}_{\widetilde{\mathrm{Q}}'} \to \mathrm{M}$ such that $\mathrm{p} = \mathrm{r} \circ \mathrm{m}$ and $\mathrm{q} = \mathrm{f} \circ \mathrm{r}$. Therefore, we can find a cubical path $r(\widetilde{Q}')$ in M. Since $q(m(\widetilde{P})) = f(p(\widetilde{P})) = f(P) \xrightarrow{d_i^l} Q' = q(\widetilde{Q}')$, then $m(\widetilde{P}) \xrightarrow{d_i^l} \widetilde{Q}'$, in virtue of item 1 of Lemma 3 for q. Consequently, $r(m(\widetilde{P})) \xrightarrow{d_i^l} r(\widetilde{Q}')$, due to item 2 of the same Lemma for r. As $\mathrm{p} = \mathrm{r} \circ \mathrm{m}$ and $\mathrm{q} = \mathrm{f} \circ \mathrm{r}$, we have $p(\widetilde{P}) = P \xrightarrow{d_i^l} r(\widetilde{Q}')$ and $f(r(\widetilde{Q}')) = q(\widetilde{Q}') = Q'$.

($\Leftarrow$) Let $\mathrm{f} = \langle f, 1_L\rangle : \mathrm{M} \to \mathrm{M}'$ be a morphism in $\mathbf{HDA}_L$ and the theorem conditions hold. We shall prove that f is $\mathbf{cP}_L$-open.

Given observations $\mathrm{M_{O_1}}$ and $\mathrm{M_{O_2}}$, a morphism $\iota_{\mathrm{l(w)}} = \langle \iota_{l(w)}, 1_L\rangle : \mathrm{M_{O_1}} \to \mathrm{M_{O_2}}$ is an *l-step* (*w-step*), if there exist maximal[2] cubical paths O_1 and O_2 in $\mathrm{M_{O_1}}$ and $\mathrm{M_{O_2}}$, respectively, such that $\iota_l(O_1) \xrightarrow{d_i^m} O_2$ ($\iota_w(O_1) \overset{(s,u,v)}{\longleftrightarrow} O_2$). It is easy to see that any morphism in $\mathbf{cP}_L$ is a finite composition of isomorphism, l-steps and w-steps.

Suppose a commuting diagram, i.e. there are morphisms $\mathrm{p} : \mathrm{M_P} \to \mathrm{M}$ and $\mathrm{q} : \mathrm{M_Q} \to \mathrm{M}'$ in $\mathbf{HDA}_L$ and a morphism $\mathrm{m} : \mathrm{M_P} \to \mathrm{M_Q}$ in $\mathbf{cP}_L$ such that $\mathrm{f} \circ \mathrm{p} = \mathrm{q} \circ \mathrm{m}$. We have to show that there is a morphism $\langle r, 1_L\rangle : \mathrm{M_Q} \to \mathrm{M}$ in $\mathbf{HDA}_L$ such that $\mathrm{p} = \mathrm{r} \circ \mathrm{m}$ and $\mathrm{q} = \mathrm{f} \circ \mathrm{r}$. Consider the proof of the case, when m is a w-step (the proofs of the cases, when m is an l-step or isomorphism, are similar). The general case follows from repeated applications of the arguments in the proofs of the above cases.

As $\mathrm{m} = \iota_{\mathrm{w}}$ is a w-step, there exist maximal cubical paths P and Q in the observations $\mathrm{M_P}$ and $\mathrm{M_Q}$, respectively, such that $\iota_w(P) \overset{(s,u,v)}{\longleftrightarrow} Q$. Moreover, we have $q(\iota_w(P)) \overset{(s,u,v)}{\longleftrightarrow} q(Q)$ in M', by Lemma 3. Since $f(p(P)) = q(\iota_w(P))$, there exists $P' \in \mathcal{CP}(\mathrm{M})$ such that $p(P) \overset{(s,u,v)}{\longleftrightarrow} P'$ and $f(P') = q(Q)$, due to the theorem conditions. Assuming $P' = p_0 \ldots p_k$ and $Q = q_0 \ldots q_k$ we put $r(q_j) = p_j$ and $r(d_{j_1}^{\alpha_1} \circ \cdots \circ d_{j_s}^{\alpha_s}(q_j)) = d_{j_1}^{\alpha_1} \circ \cdots \circ d_{j_s}^{\alpha_s}(p_j)$ for all $\alpha_r = 0, 1$,

[2] A cubical path in an observation is called *maximal* if the observation has the form of the cubical path.

$r = 1 \dots s$, $1 \leq j_1 < \dots < j_s \leq \dim q_j$, $1 \leq s \leq \dim q_j$ and $0 \leq j \leq n$. It is easy to see that $\mathrm{r} = \langle r, 1_L\rangle : \mathrm{M_Q} \to \mathrm{M}$ is a morphism in $\mathbf{HDA}_L$ and satisfies $\mathrm{p} = \mathrm{r} \circ \iota_\mathrm{w}$ and $\mathrm{q} = \mathrm{f} \circ \mathrm{r}$. Hence, f is a $\mathbf{cP}_L$-open morphism. □

At last, the coincidence of $\mathbf{cP}_L$-bisimulation and hhp-bisimulation is established.

Theorem 2. *Two HDA (with the same set L of actions) are* $\mathbf{cP}_L$*-bisimular iff they are hhp-bisimular.*

Proof. ($\Rightarrow$) Suppose a span $\mathrm{M}' \xleftarrow{\mathrm{f}'} \mathrm{M} \xrightarrow{\mathrm{f}''} \mathrm{M}''$ of $\mathbf{cP}_L$-open morphisms $\mathrm{f}' = \langle f', 1_L\rangle$ and $\mathrm{f}'' = \langle f'', 1_L\rangle$. Then, it is easy to show that a relation $\mathcal{R} = \{(f'(P), f''(P)) \mid P \in \mathcal{CP}(\mathrm{M})\}$ is an hhp-bisimulation between M' and M'', using Definition 3, Lemma 3 and Theorem 1.

($\Leftarrow$) Assume $\mathcal{R}$ to be an hhp-bisimulation between M' and M''. We have to construct a span $\mathrm{M}' \xleftarrow{\mathrm{f}'} \mathrm{M} \xrightarrow{\mathrm{f}''} \mathrm{M}''$ of $\mathbf{cP}_L$-open morphisms $\mathrm{f}' = \langle f', 1_L\rangle$ and $\mathrm{f}'' = \langle f'', 1_L\rangle$.

For $(P,Q) \in \mathcal{R}$, define $\langle P,Q\rangle = \{(P',Q') \mid P \overset{(s_1,u_1,v_1)}{\longleftrightarrow} \dots \overset{(s_l,u_l,v_l)}{\longleftrightarrow} P'$, $Q \overset{(s_1,u_1,v_1)}{\longleftrightarrow} \dots \overset{(s_l,u_l,v_l)}{\longleftrightarrow} Q'$, for some $s_m, u_m, v_m,\ 1 \leq m \leq l,\ l \geq 1\} \cup \{(P,Q)\}$.

Construct a triple $(M, i_0^\mathrm{M}, l_L^\mathrm{M})$ (denoted $\langle \mathrm{M}', \mathrm{M}''\rangle$) as follows:

- $M_n = \{\langle P,Q\rangle \mid P \in \mathcal{CP}_{p_k}(\mathrm{M}'), Q \in \mathcal{CP}_{q_k}(\mathrm{M}'')$ and $p_k \in M'_n, q_k \in M''_n\}$ with $\widehat{d_i^m}(\langle P,Q\rangle) = \langle d_i^m(P), d_i^m(Q)\rangle$ for all $\langle P,Q\rangle \in M_n$ and $n > 0$,
- $i_0^\mathrm{M} = \langle i_0^{\mathrm{M}'}, i_0^{\mathrm{M}''}\rangle$,
- $l_L^\mathrm{M}(\langle P,Q\rangle) = l_L^{\mathrm{M}'}(p_k) = l_L^{\mathrm{M}''}(q_k)$, for all $\langle P,Q\rangle \in M_1$,

We shall show that $\langle \mathrm{M}', \mathrm{M}''\rangle$ is an HDA.

Assume $\langle P,Q\rangle \in M_n$ with $n > 0$. We shall show that $\widehat{d_i^0}(\langle P,Q\rangle) \in M_{n-1}$. W.l.o.g. suppose $P = p_0 \dots p_k$ and $Q = q_0 \dots q_k$. First, we shall prove that $(P,Q) \in \mathcal{R}$ implies $(d_i^0(P)p_k, d_i^0(Q)q_k) \in \mathcal{R}$. W.l.o.g. let $d_i^0(P)$ be obtained due to the fulfillment of case (ii) in the definition of i-beginning. The corresponding adjacency-chain is $(P = P_{m+1}) \overset{m+1}{\longleftrightarrow} \dots \overset{k-1}{\longleftrightarrow} (P_k = d_i^0(P)p_k)$, or, in detail, $(P = P_{m+1}) \overset{(m+1,u_{m+1},v_{m+1})}{\longleftrightarrow} P_{m+2} \overset{(m+2,u_{m+2},v_{m+2})}{\longleftrightarrow} \dots \overset{(k-2,u_{k-2},v_{k-2})}{\longleftrightarrow} P_{k-1} \overset{(k-1,u_{k-1},v_{k-1})}{\longleftrightarrow} (P_k = d_i^0(P)p_k)$. By item 5 of Definition 4, there exist $Q_{m+2}, \dots, Q_k \in \mathcal{CP}(\mathrm{M}'')$ such that $(Q = Q_{m+1}) \overset{(m+1,u_{m+1},v_{m+1})}{\longleftrightarrow} Q_{m+2} \overset{(m+2,u_{m+2},v_{m+2})}{\longleftrightarrow} \dots \overset{(k-2,u_{k-2},v_{k-2})}{\longleftrightarrow} Q_{k-1} \overset{(k-1,u_{k-1},v_{k-1})}{\longleftrightarrow} Q_k$ and $(P_s,Q_s) \in \mathcal{R}$ for all $(m+2) \leq s \leq k$. W.l.o.g. assume $P_s = p_0^s \dots p_k^s$ and $Q_s = q_0^s \dots q_k^s$, for all $(m+2) \leq s \leq k$. Consider an arbitrary $(P_s,Q_s) \in \mathcal{R}$ with $(m+1) \leq s \leq (k-1)$. Since (s,u_s,v_s)-adjacency $P_s \overset{(s,u_s,v_s)}{\longleftrightarrow} P_{s+1}$ belongs to the adjacency-chain corresponding to i-beginning of P, P_s contains the segment $\xrightarrow{d_{u_s}^0} p_s^s \xrightarrow{d_{v_s}^{\lambda_s}}$ and, moreover, $u_s \neq v_s$ if $\lambda_s = 1$ (due to Remark 2). By Definition 4, P_s and Q_s are l-related. So, $\dim p_r^s = \dim q_r^s$ for all $0 \leq r \leq k$.

Hence, Q_s contains the segment $\xrightarrow{d^0_{u_s}} q^s_s \xrightarrow{d^{\lambda_s}_{v_s}}$. Due to Lemma 1 applied to Q_s ($(m+1) \leq s \leq (k-1)$), $Q_{m+2}, \ldots, Q_k$ are unique cubical paths in M″, and, moreover, $Q_k = q^k_0 \ldots d^0_i(q^k_k)q^k_k$. This means that the unique number $m(Q, i)$ from the proof of Lemma 2, coincides with m, and $Q_k = d^0_i(Q)q_k$. Thus, $(d^0_i(P)p_k, d^0_i(Q)q_k) = (P_k, Q_k) \in \mathcal{R}$. Using item 3 of Definition 4, we get $(d^0_i(P),\, d^0_i(Q)) \in \mathcal{R}$, i.e. $\widehat{d^0_i}(\langle P, Q\rangle) \in M_{n-1}$. Applying item 1 of Definition 4 to $(P, Q) \in \mathcal{R}$, we obtain $\widehat{d^1_i}(\langle P, Q\rangle) \in M_{n-1}$.

Following the reasoning of the proof of Lemma 4, the commutativity of the diagrams in Definition 1 is clear. The non-degeneracy property of $\langle \mathrm{M}', \mathrm{M}''\rangle$ follows from the non-degeneracy properties of M′ and M″. Thus, $\langle \mathrm{M}', \mathrm{M}''\rangle$ is an HDA.

Define mappings $\langle pr_1, 1_L\rangle : \langle \mathrm{M}', \mathrm{M}''\rangle \to \mathrm{M}'$ and $\langle pr_2, 1_L\rangle : \langle \mathrm{M}', \mathrm{M}''\rangle \to \mathrm{M}''$ as follows: $pr_1(\langle P, Q\rangle) = p$ and $pr_2(\langle P, Q\rangle) = q$ for all $\langle P, Q\rangle \in M$ with $P \in \mathcal{CP}_p(\mathrm{M}')$ and $Q \in \mathcal{CP}_q(\mathrm{M}'')$. It is routine to show that $\langle pr_1, 1_L\rangle$ and $\langle pr_2, 1_L\rangle$ are morphisms in $\mathbf{HDA}_L$. Consider the proof of $\mathbf{cP}_L$-openness of $\langle pr_1, 1_L\rangle$ (the proof of $\mathbf{cP}_L$-openness of $\langle pr_2, 1_L\rangle$ is similar). Take an arbitrary $O = o_0 \ldots o_k \in \mathcal{CP}(\langle \mathrm{M}', \mathrm{M}''\rangle)$. Then, $pr_1(O) \in \mathcal{CP}(\mathrm{M}')$ and $pr_2(O) \in \mathcal{CP}(\mathrm{M}'')$ by Lemma 3. W.l.o.g. assume $pr_1(O) = p_0 \ldots p_k$ and $pr_2(O) = q_0 \ldots q_k$. By induction on the number of the cubes in the cubical path O, it is easy to show that $o_i = \langle p_0 \ldots p_i, q_0 \ldots q_i\rangle$, for all $0 \leq i \leq k$. Hence, we have that $(pr_1(O), pr_2(O)) \in \mathcal{R}$, due to the construction of $\langle \mathrm{M}', \mathrm{M}''\rangle$. We only prove that condition 1 of Theorem 1 is true for a morphism $\langle pr_1, 1_L\rangle$ (the proof of fulfillment of condition 2 of the same theorem is similar).

Suppose $pr_1(O) \xrightarrow{d^m_i} P'$, for some $P' \in \mathcal{CP}(\mathrm{M}')$. By item 1 of Definition 4, there exists $Q' \in \mathcal{CP}(\mathrm{M}'')$ such that $pr_2(O) \xrightarrow{d^m_i} Q'$ and $(P', Q') \in \mathcal{R}$. Let $o_{k+1} = \langle P', Q'\rangle$. Due to the construction of $\langle \mathrm{M}', \mathrm{M}''\rangle$, we have $O' = o_0 \ldots o_k o_{k+1} \in \mathcal{CP}(\langle \mathrm{M}', \mathrm{M}''\rangle)$ and $O \xrightarrow{d^m_i} O'$. Obviously, $pr_1(O') = P'$. Hence, $\langle pr_1, 1_L\rangle$ is a $\mathbf{cP}_L$-open morphism, by Theorem 1. □

3 Timed HDA

3.1 The category THDA

We begin with presenting the concept of a timed HDA (THDA) [9] – a timed extension of HDA. THDA are defined as a geometric shape together with a structure given by cubes realized on this shape, and a family of norms defining the infinitesimal duration of a computation in all directions.

Introduce some auxiliary notions and notations. Consider a unit cube of dimension n in $\mathbb{R}^n$: $\square_n := \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leq t_i \leq 1,\ i = 1, \ldots, n\}$ for $n > 0$, and $\square_0 := \{0\}$ for $n = 0$. Let $\mathring{\square}_n$ denote the topological interior of $\square_n$, i.e. $\mathring{\square}_n := \{(t_1,\, \ldots,\, t_n) \in \mathbb{R}^n \mid 0 < t_i < 1,\ i = 1, \ldots, n\}$ for $n > 0$, and $\mathring{\square}_0 := \{0\}$ for $n = 0$.

In order to define a THDA we first need a geometric shape (topological space) X. We are especially interested in compactly generated Hausdorff topological spaces[3] [16]. Then we should give a differential structure on X to be able to measure time. In our case the differential structure on X is given by cubes. Intuitively, cubes should be a sort of deformed cubes, so we define them as continuous mappings $x : \Box_n \to X$ which induce homeomorphisms from $\mathring{\Box}_n$ to their images. Thus, $x : \Box_n \to X$ gives the trivial structure of manifold[4] to $x(\mathring{\Box}_n)$. For a cube $x(\mathring{\Box}_n)$ $(n > 0)$, we can define its coordinates as follows: $(x(t_1, \ldots, t_n))_i = t_i$ $(i = 1, \ldots, n)$. We consider mappings $x : \Box_n \to X$ to be continuously deformed cubes only in their interior since we may want to identify some of their boundaries to get cyclic shapes. To do this we need functions characterizing the boundaries of cubes. Assume $\delta_i^m : \Box_n \to \Box_{n+1}$ $(i \in \{1, \ldots, n+1\}, m \in \{0,1\})$ to be continuous functions defined as follows: $\delta_i^m(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, m, t_i, \ldots, t_n)$ for $n > 0$, and $\delta_1^m(0) = (m)$ for $n = 0$. We then have $\delta_i^k \delta_j^m = \delta_{j+1}^m \delta_i^k$ for $i \leq j$. To be able to take boundaries we should require the collection of cubes to be stable by composition with boundary functions. To illustrate the concepts, consider Figure 6. We have the square $\Box_2$, the edge $\Box_1$ and the torus T. Moreover, x_2 continuously maps the square $\Box_2$ into T so that $x_2(\mathring{\Box}_2)$ is a torus without the small circle $x_2(t, 0)$ $(0 \leq t \leq 1)$ and the big circle $x_2(0, t)$ $(0 \leq t \leq 1)$, and x_1 continuously maps the edge $\Box_1$ into the small circle of T so that $x_1(\mathring{\Box}_1)$ is the small circle without the intersection of the circles. Then, we get $x_1 = x_2 \circ \delta_1^0$.

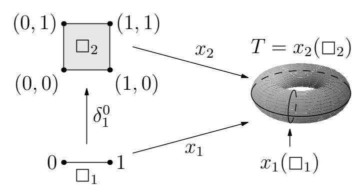


Figure 6: Taking a boundary.

We can now split our cubes into sets X_n containing only cubes with the domain $\Box_n$. Also, we require X to be covered by all its cubes, i.e. X is the disjoint union $\bigsqcup_{x \in X_n,\ n \in \mathbb{N}} (x(\mathring{\Box}_n))$.

Finally, to measure the time of cubes from X, we are to have a norm $\|\cdot\|_u$ on the tangent space $T_u X =_{def} T_u x(\mathring{\Box}_n)$ $(u \in x(\mathring{\Box}_n))$ at every $u \in X$ (for further details see [24]). A tangent space $T_u x(\mathring{\Box}_n)$ is an n-dimensional space consisting of the tangent vectors $\dot{u}$ of the curves through a point u, which can be measured

[3] In topology, a compactly generated space is a topological space X satisfying the following condition: each subspace $U \subset X$ which intersects every compact subset K of X in a closed set is itself closed.

[4] The definition of the notion of manifold can be found in [24].

by the norm. Intuitively, a tangent space contains the possible "directions" in which one can pass through u and the norm can be seen as an infinitesimal duration of the computation at u. In order to be consistent with the space, the norm $F(u, \dot{u}) = \| \dot{u} \|_u$ should be a continuous mapping for all $u \in X, \dot{u} \in T_u X$.

We are now ready to define (labelled) THDA. For full details and explanations on the definitions related to THDA, we refer the reader to [9], where the concept has been first introduced.

Definition 6. A *(labelled non-degenerate) THDA* is a tuple $\mathrm{X} = (X, i_0^{\mathrm{X}}, l_L^{\mathrm{X}}, \| \cdot \|_X)$, where

- X is a compactly generated Hausdorff topological space together with a presentation of X by singular cubes, i.e. X is the disjoint union $\bigsqcup_{x \in X_n,\ n \in \mathbb{N}} x(\mathring{\Box}_n)$, where X_n consists of continuous mappings $x^n : \Box_n \to X$ which induce homeomorphisms from $\mathring{\Box}_n$ to its image and are such that $x^n \circ \delta_i^m \in X_{n-1}$ for all $i = 1, \ldots, n$ and $m = 0, 1$. Moreover, for all $x \in X_n$ and $m = 0, 1$ we assume that the non-degeneracy property holds: $|\{x \circ \delta_i^m \mid i = 1 \ldots n\}| = n$,
- i_0^{X} is a distinguished basepoint of X called the *initial point* and represented in the form of $i_0^{\mathrm{X}} = x(0)$[5] for some mapping $x \in X_0$,
- $l_L^{\mathrm{X}} : X_1 \to L$ is a *labelling function* from the 1-cubes of X to a set L of actions such that $l_L^{\mathrm{X}}(x \circ \delta_i^0) = l_L^{\mathrm{X}}(x \circ \delta_i^1)$ for all $i = 1, 2$ and $x \in X_2$,
- X is given a family of norms $\| \cdot \|_u$ on every tangent space[6] $T_u X =_{def} T_u x(\mathring{\Box}_n)$ $(u \in x(\mathring{\Box}_n))$ such that $F(u, \dot{u}) = \| \dot{u} \|_u$ is a continuous mapping from the tangent bundle $TX =_{def} \bigsqcup_{u \in X} T_u X$ with its natural topology[7] to the half-line $\mathbb{R}^+$ with the induced topology from $\mathbb{R}$.

[5] If there is no confusion, we shall denote $i_0^{\mathrm{X}} = x \in X_0$.

[6] Suppose that two curves $\nu_{1/2} : (-1, 1) \to X$, passing through the same point $u \in x(\mathring{\Box}_n) \subseteq X$, are given such that both $x^{-1} \circ \nu_1$ and $x^{-1} \circ \nu_2$ are differentiable at 0. Then, ν_1 and ν_2 are said to be tangent at 0, if $\nu_1(0) = \nu_2(0) = u$ and the ordinary derivatives of $x^{-1} \circ \nu_1$ and $x^{-1} \circ \nu_2$ at 0 coincide. This defines an equivalence relation on such curves. The equivalence classes, denoted by $\langle \nu \rangle_u$ for a curve ν, are known as the tangent vectors of X at u. The tangent space of X at u, denoted by $T_u X$, is specified as the set of all tangent vectors. For $x \in X_n$, define a mapping $\theta_x : \mathring{\Box}_n \times \mathbb{R}^n \to \bigsqcup_{u \in x(\mathring{\Box}_n)} T_u x(\mathring{\Box}_n) = \bigsqcup_{u \in x(\mathring{\Box}_n)} \{u\} \times T_u x(\mathring{\Box}_n)$ as follows: $\theta_x(t, v) = (x(t), \langle \nu_v \rangle_{x(t)})$ for all $(t, v) \in \mathring{\Box}_n \times \mathbb{R}^n$, where $\nu_v(s) = x(t + vs)$ for all $s \in (-1, 1)$. Clearly, θ_x is a bijection, and a vector space isomorphism when restricted to each $t \times \mathbb{R}^n$. For subsequent purposes, we need the "inverse" of θ_x defined by $\theta_x^{-1}(u, \langle \nu \rangle_u) = (x^{-1}(u), \frac{d}{ds}(x^{-1} \circ \nu)(0))$ for all $(u, \langle \nu \rangle_u) \in \bigsqcup_{u \in x(\mathring{\Box}_n)} \{u\} \times T_u x(\mathring{\Box}_n)$.

[7] For a fixed base $\mathcal{B}_X$ of the topology on X, a topology on TX is defined by using its base $\mathcal{B}_{TX}$. Let $V \in \mathcal{B}_{TX}$ iff $V = \bigsqcup_{x \in X_n^U, n \geq 0} \theta_x(W_x, B_x)$. Here, $U \in \mathcal{B}_X$, $X_n^U = \{x \in X_n \mid U \cap x(\mathring{\Box}_n) \neq \emptyset\}$, θ_x is a bijection specified in Footnote 6, $W_x = x^{-1}(U \cap x(\mathring{\Box}_n))$ and

Whenever no confusion is possible we drop subscripts and superscripts on $\mathrm{X} = (X, i_0^{\mathrm{X}}, l_L^{\mathrm{X}}, \|\cdot\|_X)$ and write $\mathrm{X} = (X, i_0, l, \|\cdot\|)$ instead, to denote a THDA X over a set L of actions.

Remark 3. Assume $\mathrm{X} = (X, i_0, l, \|\cdot\|)$ to be a THDA over a set L of actions. We have $l(y \circ \delta_j^0) = l(y \circ \delta_j^1)$ for all $j = 1, 2$ and $y \in X_2$. So, to extend a labelling function to all $x \in X_n$ $(n \geq 0)$ define $l(x) = (l_1(x), \ldots, l_n(x))$ with $l_i(x) = l(x \circ \delta_n^{\varepsilon_n^i} \circ \ldots \delta_{i+1}^{\varepsilon_{i+1}^i} \circ \delta_{i-1}^{\varepsilon_{i-1}^i} \circ \ldots \circ \delta_1^{\varepsilon_1^i})$ if $n > 1$ and $l(x) = \emptyset$ if $n = 0$.

In order to know how much time cubes of a THDA may take, we introduce the following definition of paths as being particular curves between two points in X. A continuous mapping $\gamma : [0,1] \to X$ is called a *path* in a THDA X if there exist open intervals $I_k = (\tau_{k-1}, \tau_k)$ and cubes $x_k \in X_{n_k}$ $(1 \leq k \leq m)$ such that $\tau_0 = 0$, $\tau_m = 1$ and for every $1 \leq k \leq m$ the following conditions hold: the mapping $\gamma : I_k \to x_k(\overset{\circ}{\Box}_{n_k})$ is non-decreasing, w.r.t. each coordinate in the cube x_k, and the mapping $x_k^{-1} \circ \gamma : I_k \to \overset{\circ}{\Box}_{n_k}$ is differentiable for $n_k > 0$. The *length* of a path γ is calculated as follows: $length(\gamma) = \int_0^1 \|\dot{\gamma}(s)\|_{\gamma(s)} ds$[8], where $\dot{\gamma}(s)$ is given by $(\gamma(s), \dot{\gamma}(s)) = \theta_{x_k}(x_k^{-1}(\gamma(s)), \frac{d}{ds}(x_k^{-1} \circ \gamma)(s))$ for $s \in I_k$ $(1 \leq k \leq m)$ (see Footnote 6).

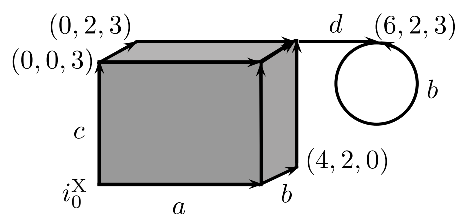
X :


Figure 7: An example of a THDA X.

Example 5. Figure 7 shows a trivial example of a THDA. The THDA $\mathrm{X} = (X = x(\Box_3) \cup x_1(\Box_1) \cup x_0(\Box_1),\ i_0^{\mathrm{X}},\ l_L^{\mathrm{X}},\ \|\cdot\|_X)$ is generated by the 3-cube $x(t_1, t_2, t_3) = (4t_1, 2t_2, 3t_3)$ $((t_1, t_2, t_3) \in \Box_3)$, the 1-cube $x_1(t) = (4 + 2t, 2, 3)$ $(t \in \Box_1)$ and the 1-cube $x_0(t) = (6 - \sin(2\pi t), 2, 2 + \cos(2\pi t))$ $(t \in \Box_1)$ which is depicted by the filled-in cube, the segment and the circle, respectively. The initial point is $i_0^{\mathrm{X}} = (0, 0, 0)$. Having a set $L = \{a, b, c, d\}$, the labelling function is given by $l_L^{\mathrm{X}}(x \circ \delta_3^0 \circ \delta_2^0) = a$, $l_L^{\mathrm{X}}(x \circ \delta_3^0 \circ \delta_1^0) = b$, $l_L^{\mathrm{X}}(x \circ \delta_2^0 \circ \delta_1^0) = c$, $l_L^{\mathrm{X}}(x_1) = d$ and $l_L^{\mathrm{X}}(x_0) = b$. The norm $\|\cdot\|_X$ is induced by the Euclidean one in $\mathbb{R}^3$. Notice that geometrically, the interior of the filled-in cube consists of the union of all paths where occurrences of a, b and c overlap in time. The lengths of the

B_x is an open ball in $\mathbb{R}^n$ such that $x_1 = x \circ \delta_k^m$ implies $B_{x_1} = pr_k B_x$ for $x_1 \in X_{n-1}^U$, where $pr_k : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is a projection defined by $pr_k(t_1, \ldots, t_n) = (t_1, \ldots, \widehat{t_k}, \ldots, t_n)$ for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$.

[8]The integral is actually the sum of the integrals over intervals I_k $(1 \leq k \leq m)$.

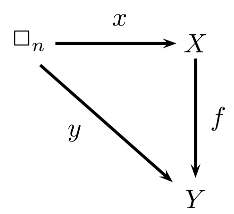


Figure 8: A diagram relating a cube $x \in X_n$ to a cube $y \in Y_n$ via a mapping f.

paths travelled along the 1-cube labelled by a (b or c) are equal to 4 (2 or 3, respectively). Then, in the filled-in cube, the lengths of all paths starting with $(0, 0, 0)$ and ending with $(4, 2, 3)$ vary from $\sqrt{4^2 + 2^2 + 3^2}$ to $4 + 2 + 3$.

Consider the definition of a morphism mapping points and actions of the simulated system to simulating points and actions of the other and satisfying some requirements. Note, we want morphisms to contract time.

Definition 7. Let $\mathrm{X} = (X, i_0^{\mathrm{X}}, l_{L^{\mathrm{X}}}^{\mathrm{X}}, \|\cdot\|_X)$ and $\mathrm{Y} = (Y, i_0^{\mathrm{Y}}, l_{L^{\mathrm{Y}}}^{\mathrm{Y}}, \|\cdot\|_Y)$ be THDA. A mapping $\mathrm{f} = \langle f, \alpha \rangle$ (where $f : X \to Y$ is a continuous mapping, $\alpha : L^{\mathrm{X}} \to L^{\mathrm{Y}}$ is a mapping) is called a *morphism* from X to Y iff the following holds:

1. $f(i_0^{\mathrm{X}}) = i_0^{\mathrm{Y}}$,
2. for any mapping $x \in X_n$ ($n \in \mathbb{N}$), there exists a mapping $y \in Y_n$ such that
   a) the diagram in Figure 8 commutes,
   b) $l_{L^{\mathrm{Y}}}^{\mathrm{Y}}(y) = \alpha(l_{L^{\mathrm{X}}}^{\mathrm{X}}(x))$,
3. $\|d_u f(\dot{u})^9\|_{f(u)} \leq \|\dot{u}\|_u$ for all $\dot{u} \in T_u X$ and $u \in X$.

The first condition guarantees that a morphism preserves initial points. The second ensures that a morphism maps an n-cube in X to an n-cube in Y, respecting their labellings. The third condition says that the length of each path in X is not less than the length of its image. If in the third condition we have $\|d_u f(\dot{u})\|_{f(u)} = \|\dot{u}\|_u$ for all $\dot{u} \in T_u X$ and $u \in X$, then f preserves the length of every path (i.e. f is an isometry).

THDA with morphisms between them form a category $\mathbf{THDA}_{\leq}$ in which the composition of two morphisms $\mathrm{f} = \langle f, \alpha \rangle : \mathrm{X} \to \mathrm{Y}$ and $\mathrm{g} = \langle g, \beta \rangle : \mathrm{Y} \to \mathrm{Z}$ is $\mathrm{g} \circ \mathrm{f} = \langle g \circ f,\ \beta \circ \alpha \rangle : \mathrm{X} \to \mathrm{Z}$, and the identity morphism is a pair of the identity mappings.

[9] A mapping $df : TX \to TY$ defined as follows: for $u \in x(\mathring{\square}_n)$, $f(u) \in y(\mathring{\square}_n)$ and $\dot{u} \in T_u X$, $df(u, \dot{u}) (=_{def}\ d_u f(\dot{u})) = \theta_y \circ d(y^{-1} \circ f \circ x) \circ \theta_x^{-1}(u, \dot{u})$, is the differential of f. Here, the differential (in a usual sense) $d(y^{-1} \circ f \circ x)$ of $y^{-1} \circ f \circ x : \mathring{\square}_n \to \mathring{\square}_n$, equals to the identity mapping, due to the commutativity of the diagram in Figure 8. This means that $df(u, \dot{u}) = \theta_y \circ \theta_x^{-1}(u, \dot{u})$.

3.2 Relating HDA and THDA

In his thesis [9], Goubault has proposed adjoint functors $\mathcal{T} : \mathbf{HDA} \to \mathbf{THDA}$ and $\mathcal{F}t : \mathbf{THDA} \to \mathbf{HDA}$ between the category **HDA** and the category **THDA**. The objects of **THDA** are THDA from Definition 6 and the morphisms are mappings from Definition 7 but satisfying only items 1 and 2.

We shall adapt the functors for use in our categories **HDA** and $\mathbf{THDA}_{\leq}$.

Proposition 1.

1. *Define a mapping* $\mathcal{T} : \mathbf{HDA} \to \mathbf{THDA}_{\leq}$ *on objects* $(M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})$ *as follows:* $\mathcal{T}((M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})) = (X, i_0^{\mathrm{X}}, l_L^{\mathrm{X}}, \|\cdot\|_X)$*, where*

   - $X = \bigsqcup_{x \in M_n, n \geq 0} (x, \Box_n)/\!\equiv$ *with the quotient space topology induced by* $\bigsqcup_{x \in M_n, n \geq 0} (x, \Box_n)$ *with the disjoint sum topology, where every* $(x, \Box_n)$ *inherits the standard topology on* $\mathbb{R}^n$*. Here, the equivalence* $\equiv$ *is defined by* $(d_i^m(x), \Box_{n-1}) \equiv (x, \delta_i^m(\Box_{n-1}))$*. Set* $X_n = \{(x, \cdot) : \Box_n \to X \mid x \in M_n\}$*;*
   - $i_0^{\mathrm{X}} = (i_0^{\mathrm{M}}, \Box_0)$*;*
   - $l_L^{\mathrm{X}}(x, \cdot) = l_L^{\mathrm{M}}(x)$*, for all* $x \in M_1$*;*
   - $\|\dot{u}\|_{(x,t)} = \max_{1 \leq i \leq n} |u_i|$[10]*, for all* $\dot{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n = T_{(x,t)}(x, \mathring{\Box}_n)$*,* $t \in \mathring{\Box}_n$ *and* $x \in M_n$*,*

   and on morphisms $\langle g, \alpha \rangle : \mathrm{M}^1 \to \mathrm{M}^2$ *as follows:* $\mathcal{T}(\langle g, \alpha \rangle) = \langle \hat{g}, \alpha \rangle$*, where* $\hat{g}(x, t) = (g(x), t)$ *for all points* (x, t) *of* $\mathcal{T}(\mathrm{M}^1)$*. Then,* $\mathcal{T}$ *is a functor called a geometric realization functor.*

2. *Define a mapping* $\mathcal{F}t : \mathbf{THDA}_{\leq} \to \mathbf{HDA}$ *on objects* $(X, i_0^{\mathrm{X}}, l_L^{\mathrm{X}}, \|\cdot\|_X)$ *as follows:* $\mathcal{F}t((X, i_0^{\mathrm{X}}, l_L^{\mathrm{X}}, \|\cdot\|_X)) = (M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})$*, where*

   - $M_n = X_n$ *with* $d_i^m(x) = x \circ \delta_i^m$ *for all* $x \in X_n$ *and* $n \geq 1$*;*
   - $i_0^{\mathrm{M}} = x_0$*, where* $x_0(0) = i_0^{\mathrm{X}}$*;*
   - $l_L^{\mathrm{M}} = l_L^{\mathrm{X}}$*,*

   and on morphisms $\langle f, \alpha \rangle : \mathrm{X}^1 \to \mathrm{X}^2$ *as follows:* $\mathcal{F}t(\langle f, \alpha \rangle) = \langle \check{f}, \alpha \rangle$*, where* $\check{f}(x) = f \circ x$*, for all cube* x *of* $\mathcal{F}t(\mathrm{X}^1)$*. Then,* $\mathcal{F}t$ *is a functor called a forgetting functor.*

Proof. Since the morphisms in $\mathbf{THDA}_{\leq}$ differ from the morphisms in **THDA** by the presence of item 3 in Definition 7, it is sufficient to show that $\langle \hat{g}, \alpha \rangle = \mathcal{T}(\langle g, \alpha \rangle)$ ($\langle g, \alpha \rangle$ is a morphism in **HDA**) satisfies item 3 of Definition 7. But it is obvious because inequality $\|d_{(x,t)}\hat{g}(\dot{t})\|_{\hat{g}((x,t))} \leq \|\dot{t}\|_{(x,t)}$ turns into equality as the vectors are the same and the both norms are Chebyshev. ∎

[10] This norm is called Chebyshev norm.

In contrast to [9], it has turned out that the functors $\mathcal{T}$ and $\mathcal{F}t$ between the categories **HDA** and $\mathbf{THDA}_{\leq}$ are not adjoint. Nevertheless, we shall show that timed versions of Theorems 1 and 2 hold. For this purpose, we need the following auxiliary notion and facts.

Let X be a topological space satisfying the first item of Definition 6. Then, X is called a $\Box$-*topological space* if the topology on X coincides with a topology defined as follows: U is open in X iff $x^{-1}(U)$ is open in $\Box_n$[11] for all $x \in X_n$ and $n \geq 0$.

Lemma 5. *Let X and Y be topological spaces satisfying the first item of Definition 6 and $f : X \to Y$ be a mapping meeting item 2a) of Definition 7. If X is a $\Box$-topological space, then $f : X \to Y$ is a continuous mapping. Moreover, $df : TX \to TY$ is a continuous mapping as well.*

Proof. First, we shall prove that $f : X \to Y$ is a continuous mapping. Take an arbitrary open set V in Y. We have to show that $f^{-1}(V)$ is open in X. A set $y^{-1}(V)$ is open in $\Box_n$, for all $y \in Y_n$ $(n \geq 0)$, because any $y : \Box_n \to Y$ is a continuous mapping. In particular, $x^{-1} \circ f^{-1}(V)$ is open in $\Box_n$, for all $x \in X_n$ $(n \geq 0)$. Due to X being a $\Box$-topological space, $f^{-1}(V)$ is open in X.

Next, we shall show that $df : TX \to TY$ is a continuous mapping. Fix bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ of the topologies on X and Y, respectively. Take an arbitrary set $\widetilde{V}$ from the base $\mathcal{B}_{TY}$ of the topology on TY, i.e. $\widetilde{V} = \bigsqcup_{y \in Y_n^V, n \geq 0} \theta_y(W_y, B_y)$ with $V \in \mathcal{B}_Y$, $W_y = y^{-1}(V \cap y(\mathring{\Box}_n))$ and B_y is an open ball in $\mathbb{R}^n$ such that $y_1 = y \circ \delta_k^m$ implies $B_{y_1} = pr_k B_y$ (see Footnote 7). We need to prove that $(df)^{-1}(\widetilde{V})$ is an open set in TX. We have

$$\begin{aligned}(df)^{-1}(\widetilde{V}) &= \bigsqcup_{y \in Y_n^V, n \geq 0} (df)^{-1}(\theta_y(W_y, B_y)) = \\ &= \bigsqcup_{y \in Y_n^V, n \geq 0} \left(\bigsqcup_{x \in \{x \mid y = f \circ x\}} \theta_x(\theta_y^{-1}(\theta_y(W_y, B_y))) \right) = \\ &= \bigsqcup_{y \in Y_n^V, n \geq 0} \left(\bigsqcup_{x \in \{x \mid y = f \circ x\}} \theta_x(W_y, B_y) \right).\end{aligned}$$

Since $f : X \to Y$ is a continuous mapping and $V \in \mathcal{B}_Y$,

$$\begin{aligned}f^{-1}(V) &= f^{-1}\left(\bigsqcup_{y \in Y_n^V, n \geq 0} y(W_y) \right) = \bigsqcup_{y \in Y_n^V, n \geq 0} f^{-1}(y(W_y)) = \\ &= \bigsqcup_{y \in Y_n^V, n \geq 0} \left(\bigsqcup_{x \in \{x \mid y = f \circ x\}} x(W_y) \right) = U\end{aligned}$$

is an open set in X. By the definition of a base of a topology, we get $U = \cup_\alpha U_\alpha$, where $U_\alpha \in \mathcal{B}_X$. As $U_\alpha \subseteq X$, it holds that

$$U_\alpha = \bigsqcup_{x_\alpha \in X_n^{U_\alpha}, n \geq 0} x_\alpha(G_{x_\alpha}),$$

[11] The topology on $\Box_n \subseteq \mathbb{R}^n$ is induced by the Euclidean space $\mathbb{R}^n$.

where G_{x_α} is an open set in $\overset{\circ}{\Box}_n$. So, $(df)^{-1}(\widetilde{V}) = \cup_\alpha \widetilde{U}_\alpha$ with

$$\widetilde{U}_\alpha = \bigsqcup_{x_\alpha \in X_n^{U_\alpha}, n \geq 0} \theta_{x_\alpha}(G_{x_\alpha}, B_{(f \circ x_\alpha)}) \in \mathcal{B}_{TX}.$$

Thus, $(df)^{-1}(\widetilde{V})$ is an open set in TX. □

Lemma 6. *Let* $\mathrm{M} = (M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})$ *and* $\mathrm{Y} = (Y, i_0^{\mathrm{Y}}, l_{L^{\mathrm{Y}}}^{\mathrm{Y}}, \|\cdot\|_Y)$ *be objects in* **HDA** *and* $\mathbf{THDA}_{\leq}$*, respectively, and* $\mathrm{f} = \langle f, \alpha \rangle : \mathrm{M} \to \mathcal{F}t(\mathrm{Y})$ *be a morphism in* **HDA**. *Then, a structure* $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M}) = \mathrm{X} = (X, i_0^{\mathrm{X}}, l_L^{\mathrm{X}}, \|\cdot\|_X)$ *with* X*,* i_0^{X} *and* l_L^{X} *specified as in Proposition 1 and* $\|\cdot\|_X$ *defined as follows:* $\|\cdot\|_{(x,t)} = \|d_{(x,t)}\hat{f}(\cdot)\|_{\hat{f}(x,t)}$ *with* $\hat{f}((x,t)) = f(x)(t)$ *for all* $(x,t) \in X$*, is an object and* $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{f}) = \langle \hat{f}, \alpha \rangle : \mathcal{T}_{\mathrm{f,Y}}(\mathrm{M}) \to \mathrm{Y}$ *is a morphism in* $\mathbf{THDA}_{\leq}$.

Proof. By Proposition 1, X satisfies all the conditions, except for the last one, of Definition 6. Let us prove that X is a $\Box$-topological space. Consider a mapping $(x, \cdot) : \Box_n \to X$. Clearly, it coincides with the composition $\phi \circ \iota_x \circ \sigma_x$, where $\sigma_x : \Box_n \to (x, \Box_n)$ is the identical map, $\iota_x : (x, \Box_n) \to \bigsqcup_{x \in M_n, n \geq 0} (x, \Box_n)$ is the inclusion map, and $\phi : \bigsqcup_{x \in M_n, n \geq 0} (x, \Box_n) \to X$ is the quotient map. By the definition of the topologies, U is open in X iff $\phi^{-1}(U)$ is open in $\bigsqcup_{x \in M_n, n \geq 0} (x, \Box_n)$ iff $\iota_x^{-1}(\phi^{-1}(U))$ is open in $(x, \Box_n)$ for all $x \in M_n$ $(n \geq 0)$ iff $\sigma_x^{-1}(\iota_x^{-1}(\phi^{-1}(U)))$ is open in $\Box_n$ for all $x \in M_n$ $(n \geq 0)$, i.e. $(x, \cdot)^{-1}(U)$ is open in $\Box_n$ for all $(x, \cdot) \in X_n$ $(n \geq 0)$. Hence, X is a $\Box$-topological space. By the definition of $\hat{f}$, condition 2a) of Definition 7 holds as well. By Lemma 5, $\hat{f}$ is a continuous mapping. Clearly, the mapping $\langle \hat{f}, \alpha \rangle : \mathrm{X} \to \mathrm{Y}$ meets conditions 1,2b) and 3 of Definition 7. Despite this fact, we can not regard it as a morphism in $\mathbf{THDA}_{\leq}$ unless we prove that X is a THDA. It remains to show that the norm $\|\cdot\|_X$ is continuous on TX. Due to the construction of X, we have $\|\cdot\|_X = \|\cdot\|_Y \circ d\hat{f}$. By Lemma 5, $d\hat{f}$ is a continuous mapping. Using the continuity of $\|\cdot\|_Y$, the norm $\|\cdot\|_X$ is also a continuous mapping, as it is the composition of the continuous mappings. Thus, X is an object and $\langle \hat{f}, \alpha \rangle : \mathrm{X} \to \mathrm{Y}$ is a morphism in $\mathbf{THDA}_{\leq}$. □

Lemma 7. *Let* M *be an object in* $\mathbf{HDA}_L$ *and* $\mathrm{f} : \mathrm{M} \to \mathcal{F}t(\mathrm{Y})$ *be a morphism in* **HDA**. *Then*

1. $\mathrm{M} \cong_{\mathbf{HDA}_L} \mathcal{F}t(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M}))$, *i.e. there exists a morphism* $\varphi_{\mathrm{f}} : \mathrm{M} \to \mathcal{F}t(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M}))$ *and a morphism* $\psi_{\mathrm{f}} : \mathcal{F}t(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M})) \to \mathrm{M}$ *in* $\mathbf{HDA}_L$ *such that* $\psi_{\mathrm{f}} \circ \varphi_{\mathrm{f}} = id_{\mathrm{M}}$ *and* $\varphi_{\mathrm{f}} \circ \psi_{\mathrm{f}} = id_{\mathcal{F}t(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M}))}$,
2. $\mathrm{f} = \mathcal{F}t(\mathcal{T}_{f,Y}(\mathrm{f})) \circ \varphi_{\mathrm{f}}$.

Proof. Assume that M is an object in $\mathbf{HDA}_L$ and $\mathrm{f} = \langle f, \alpha \rangle : \mathrm{M} \to \mathcal{F}t(\mathrm{Y})$ is a morphism in **HDA**. Consider the proof of item 1. Define mappings $\varphi_{\mathrm{f}} = \langle \varphi_f, 1_L \rangle : \mathrm{M} \to \mathcal{F}t(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M}))$ and $\psi_{\mathrm{f}} = \langle \psi_f, 1_L \rangle : \mathcal{F}t(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M})) \to \mathrm{M}$ by $\varphi_f(x) = (x, \cdot)$, for all $x \in M$, and $\psi_f(x, \cdot) = x$, for all cube $(x, \cdot)$ from

$\mathcal{F}t(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M}))$, respectively. Clearly, these mappings are mutually inverse morphisms in $\mathbf{HDA}_L$. Next, contemplate the proof of item 2. Due to Proposition 1 and Lemma 6, $\mathcal{F}t(\mathcal{T}_{f,Y}(\mathrm{f})) = \langle \check{\hat{f}}, \alpha\rangle : \mathcal{F}t(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M})) \to \mathcal{F}t(\mathrm{Y})$ is a morphism in **HDA**. It is sufficient to show that $f = \check{\hat{f}} \circ \varphi_f$. We have $f(x) = \check{\hat{f}}(x, \cdot) = \check{\hat{f}} \circ \varphi_f(x)$ for all $x \in M$. Hence, $\mathrm{f} = \mathcal{F}t(\mathcal{T}_{f,Y}(\mathrm{f})) \circ \varphi_{\mathrm{f}}$. □

3.3 Timed hereditary history preserving bisimulation

The functor $\mathcal{F}t$ allows one to forget that the cubes in $\mathcal{F}t(\mathrm{X})$ are continuous mappings in X and to consider the cubes as a discrete set. Then, the definitions of a cubical path, an acyclic cubical path, an extension of cubical paths, s- and $(s, u, v)-$adjacency, homotopy for (discrete) HDA can be easily adapted for (continuous) THDA using $p \circ \delta_i^m$ instead of $d_i^m(p)$. If P is a cubical path in a THDA X, we shall use $P_{\mathcal{F}t}$ to denote the corresponding cubical path in the HDA $\mathcal{F}t(\mathrm{X})$. Further, $\mathcal{CP}(\mathrm{X})$ ($\mathcal{CP}_x(\mathrm{X})$) is the set of all cubical paths (ending with a cube x) in X. A point u in a THDA X is called *reachable* if there exists some $P \in \mathcal{CP}_x(\mathrm{X})$ and $u \in x(\mathring{\Box}_n)$, where $x \in X_n$. Analogously to HDA, for a cubical path $P = p_0 \dots p_k$ in a THDA $\mathrm{X} = (X, i_0, l_L, \|\cdot\|_X)$, we can define the structure $\mathrm{X}' = (X', i_0', l_L', \|\cdot\|_{X'})$, where

- $X' = \bigsqcup_{x \in (X')_n, n \geq 0} x(\mathring{\Box}_n) \subseteq X$ with the subset topology. Here, $(X')_n = \{p_i \circ \delta_{i_l}^{\alpha_l} \circ \dots \circ \delta_{i_1}^{\alpha_1} \mid \alpha_j = 0, 1, 1 \leq j \leq l, 1 \leq i_1 < \dots < i_l \leq \dim p_i, 1 \leq l \leq \dim p_i, 1 \leq i \leq k\} \cup \{p_i \mid 0 \leq i \leq k\}$,
- $i_0' = i_0$,
- $l_L' = l_L|_{(X')_1}$,
- $\|\cdot\|_{X'}$ is induced by $\|\cdot\|_X$ using the inclusion $X' \subseteq X$.

It is easy to verify that X′ is a THDA, and, moreover, a sub-THDA of X. In this case, X′ is said to *have the form of the cubical path P* in the THDA X.

We next establish that the morphisms in $\mathbf{THDA}_\leq$ represent some notions of simulation of the behaviour of one system by the other.

Lemma 8. *Given a morphism* $\mathrm{f} = \langle f, \alpha\rangle : \mathrm{X} \to \mathrm{Y}$ *in* $\mathbf{THDA}_\leq$, *for all* $P = p_0 \xrightarrow{\delta_{i_1}^{\epsilon_1}} \dots \xrightarrow{\delta_{i_k}^{\epsilon_k}} p_k \in \mathcal{CP}(\mathrm{X})$ *it holds:*

1. *there exists a unique* $f(P) = (f \circ p_0) \xrightarrow{\delta_{i_1}^{\epsilon_1}} \dots \xrightarrow{\delta_{i_k}^{\epsilon_k}} (f \circ p_k) \in \mathcal{CP}(\mathrm{Y})$;
2. *whenever* $P \xrightarrow{\delta_i^m} P'$ *in* X, *then* $f(P) \xrightarrow{\delta_i^m} f(P')$ *in* Y;
3. *whenever* $P \overset{(s,u,v)}{\longleftrightarrow} P'$ *in* X, *then* $f(P) \overset{(s,u,v)}{\longleftrightarrow} f(P')$ *in* Y;
4. $\|d_u f(\dot{u})\|_{f(u)} \leq \|\dot{u}\|_u$ *for all* $\dot{u} \in T_u p_j(\mathring{\Box}_{\dim p_j})$, $u \in p_j(\mathring{\Box}_{\dim p_j})$, $j = 1 \dots k$.

Proof. Obvious. □

Further, we extend the notion of hhp-bisimulation to THDA as follows.

Definition 8. Let X and Y be THDA.

Cubical paths $P = p_0 \dots p_k$ in X and $Q = q_0 \dots q_k$ in Y are called *d-related* iff for all $1 \le j \le k$ it holds: $\|d_t p_j(\dot{t})\|_{p_j(t)} = \|d_t q_j(\dot{t})\|_{q_j(t)}$ for all $\dot{t} \in T_t \overset{\circ}{\square}_{\dim p_j}$ and $t \in \overset{\circ}{\square}_{\dim p_j}$.

A binary relation $\mathcal{R}$ on cubical paths in X and Y is called a timed *hhp-bisimulation* between X and Y iff $\mathcal{R}_{\mathcal{F}t} = \{(P_{\mathcal{F}t}, Q_{\mathcal{F}t}) \mid (P,Q) \in \mathcal{R}\}$ is an hhp-bisimulation between $\mathcal{F}t(\mathrm{X})$ and $\mathcal{F}t(\mathrm{Y})$, and for any $(P,Q) \in \mathcal{R}$, P and Q are d-related.

THDA X and Y are *timed hhp-bisimilar* if there exists a timed hhp-bisimulation between them which relates their initial points (regarded as cubical paths).

Clearly, timed hhp-bisimulation is indeed an equivalence relation.

Example 6. Consider Figure 4. At the left side, we can see a graphical representation of the THDA $\mathrm{X} = (X = x_1(\square_2) \cup x_2(\square_2) \cup p_5(\square_1) \cup p_6(\square_1) \cup p_7(\square_1) \cup p_8(\square_1),\ s,\ l^{\mathrm{X}}_{L^{\mathrm{X}}},\ \|\cdot\|_X)$. The space X is generated by the 2-cubes: $x_1(t_1,t_2) = (-t_1,t_2)$, $x_2(t_1,t_2) = (t_1,t_2)$ $((t_1,t_2) \in \square_2)$ and the 1-cubes: $p_5(t) = (-1, 1+t)$, $p_7(t) = (-1-t, 2)$, $p_6(t) = (1, 1+t)$ and $p_8(t) = (1+t, 2)$ $(t \in \square_1)$, and has the subspace topology induced by $\mathbb{R}^2$. The initial point is $s = (0,0)$. We assume $L^{\mathrm{X}} = \{a,b,c\}$ and the labelling function $l^{\mathrm{X}}_{L^{\mathrm{X}}}$ is given by $l^{\mathrm{X}}_{L^{\mathrm{X}}}(x_1 \circ \delta^0_1) = l^{\mathrm{X}}_{L^{\mathrm{X}}}(p_1) = a$, $l^{\mathrm{X}}_{L^{\mathrm{X}}}(x_1 \circ \delta^1_2) = l^{\mathrm{X}}_{L^{\mathrm{X}}}(p_3) = b$, $l^{\mathrm{X}}_{L^{\mathrm{X}}}(x_2 \circ \delta^0_1) = l^{\mathrm{X}}_{L^{\mathrm{X}}}(p_2) = a$, $l^{\mathrm{X}}_{L^{\mathrm{X}}}(p_5) = l^{\mathrm{X}}_{L^{\mathrm{X}}}(p_6) = l^{\mathrm{X}}_{L^{\mathrm{X}}}(p_7) = l^{\mathrm{X}}_{L^{\mathrm{X}}}(p_8) = c$. The norm $\|\cdot\|_X$ is induced by the Euclidean one in $\mathbb{R}^2$. Next, at the right side, we can see a graphical representation of the THDA $\mathrm{Y} = (Y = y(\square_2) \cup q_3(\square_1) \cup q_4(\square_1) \cup q_5(\square_1) \cup q_6(\square_1),\ r,\ l^{\mathrm{Y}}_{L^{\mathrm{Y}}},\ \|\cdot\|_Y)$. The space Y is generated by the 2-cube $y(t_1,t_2) = (t_1, \lambda t_2)$ $((t_1,t_2) \in \square_2)$ and the 1-cubes: $q_3(t) = (1, \lambda + t)$, $q_4(t) = (1+t, 1+\lambda)$, $q_5(t) = (1+t, \lambda)$ and $q_6(t) = (2, \lambda + t)$ $(t \in \square_1)$ for some λ such that $1 \le \lambda \le 2$, and has the subspace topology induced by $\mathbb{R}^2$. The initial point is $r = (0,0)$. We assume $L^{\mathrm{Y}} = \{a,b,c\}$. The labelling function $l^{\mathrm{Y}}_{L^{\mathrm{Y}}}$ is given by $l^{\mathrm{Y}}_{L^{\mathrm{Y}}}(y \circ \delta^0_1) = l^{\mathrm{Y}}_{L^{\mathrm{Y}}}(q_1) = a$, $l^{\mathrm{Y}}_{L^{\mathrm{Y}}}(y \circ \delta^1_2) = l^{\mathrm{Y}}_{L^{\mathrm{Y}}}(q_2) = b$, $l^{\mathrm{Y}}_{L^{\mathrm{Y}}}(q_3) = l^{\mathrm{Y}}_{L^{\mathrm{Y}}}(q_4) = l^{\mathrm{Y}}_{L^{\mathrm{Y}}}(q_5) = l^{\mathrm{Y}}_{L^{\mathrm{Y}}}(q_6) = c$. The norm $\|\cdot\|_Y$ is induced by the Euclidean one in $\mathbb{R}^2$. It is easy to see that the THDA X and Y are timed hhp-bisimilar, if $\lambda = 1$ (take a timed hhp-bisimulation $\mathcal{R}$ as specified in example 4). In the other cases, X and Y are not timed hhp-bisimilar because the cubical path $P = sp$ in X could be related only to the cubical path $Q = rq$ in Y but P and Q are not d-related cubical paths as long as $\|d_t p(\dot{t})\|_{p(t)} = \|\dot{t}\|_t \neq \lambda \|\dot{t}\|_t = \|d_t q(\dot{t})\|_{q(t)}$ with $1 < \lambda \le 2$.

3.4 Open Maps Characterization

In this subsection, we show that timed hhp-bisimulation can be characterized by using the open maps based framework.

To deal with open maps we need to choose an observation subcategory of the category $\mathbf{THDA}_{\le}$. For a THDA X, an *observation* is a THDA $\mathrm{X_P}$ having

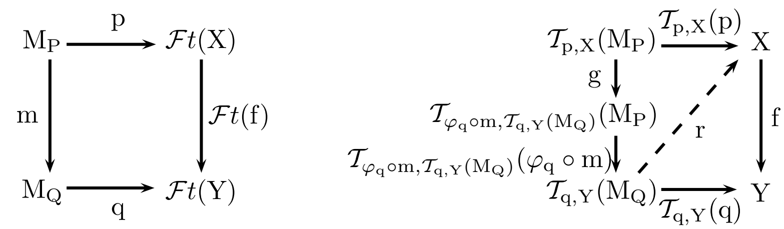


Figure 9: Diagrams for the morphism $\mathcal{F}t(\mathrm{f})$ in $\mathbf{HDA}_L$ and for the morphism f in $\mathbf{THDA}_{\leq,L}$.

the form of an acyclic cubical path P in the THDA X, and a $\square$-topological space X_P. We use $\mathbf{TcP}_{\leq}$ to denote the full subcategory of observations of the category $\mathbf{THDA}_{\leq}$.

Consider the auxiliary facts.

Lemma 9. *Let* $\mathrm{M_P}$ *be an object in* $\mathbf{cP}_L$ *and* $\mathrm{f} = \langle f, \alpha \rangle : \mathrm{M_P} \to \mathcal{F}t(\mathrm{Y})$ *be a morphism in* $\mathbf{HDA}_L$. *Then* $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M_P})$ *is an object in* $\mathbf{TcP}_{\leq,L}$.

Proof. Let $\mathrm{M_P}$ have the form of a cubical path $P = p_0 \dots p_k$ in an HDA M. Then, $\mathrm{M_P}$ has the form of the cubical path P in $\mathrm{M_P}$. Using Lemma 6, $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M_P})$ is a THDA over L. Clearly, $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M_P})$ has the form of the acyclic cubical path $\widetilde{P} = (p_0, \cdot) \dots (p_k, \cdot)$ in $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M_P})$. In the proof of Lemma 6 it has been shown that the topological space of $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M_P})$ is a $\square$-topological space. Thus, $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M_P})$ is an object in $\mathbf{TcP}_{\leq,L}$. ☐

Lemma 10. *Let* $\mathrm{X_P}$ *be an object in* $\mathbf{TcP}_{\leq,L}$, *then* $\mathcal{F}t(\mathrm{X_P})$ *is an object in* $\mathbf{cP}_L$.

Proof. Obvious. ☐

Having the category $\mathbf{THDA}_{\leq,L}$ and the accompanying subcategory $\mathbf{TcP}_{\leq,L}$, we can reason about $\mathbf{TcP}_{\leq,L}$-open morphisms and $\mathbf{TcP}_{\leq,L}$-bisimulation between objects in the category $\mathbf{THDA}_{\leq,L}$. We shall demonstrate that the functor $\mathcal{F}t$ preserves open morphisms.

Proposition 2. *Given a* $\mathbf{TcP}_{\leq,L}$*-open morphism* $\mathrm{f} : \mathrm{X} \to \mathrm{Y}$, *a morphism* $\mathcal{F}t(\mathrm{f})$ *is* $\mathbf{cP}_L$*-open.*

Proof. Suppose that the diagram shown on the left side of Figure 9 commutes. Here, $\mathrm{m} = \langle m, 1_L \rangle : \mathrm{M_P} \to \mathrm{M_Q}$ is a morphism in $\mathbf{cP}_L$ and $\mathrm{p} = \langle p, 1_L \rangle : \mathrm{M_P} \to \mathcal{F}t(\mathrm{X})$, $\mathrm{q} = \langle q, 1_L \rangle : \mathrm{M_Q} \to \mathcal{F}t(\mathrm{Y})$ are morphisms in $\mathbf{HDA}_L$. Due to Lemma 6, we can construct the diagram shown on the right side of Figure 9 with the morphisms $\mathcal{T}_{\mathrm{p,X}}(\mathrm{p}) = \langle \hat{p}, 1_L \rangle$, $\mathcal{T}_{\mathrm{q,Y}}(\mathrm{q}) = \langle \hat{q}, 1_L \rangle$ and $\mathcal{T}_{\varphi_q \circ m, \mathcal{T}_{q,Y}(M_Q)}(\varphi_q \circ \mathrm{m}) = \langle \widehat{\varphi_q \circ m}, 1_L \rangle$ in $\mathbf{THDA}_{\leq,L}$, where $\varphi_q : \mathrm{M_Q} \to \mathcal{F}t(\mathcal{T}_{\mathrm{q,Y}}(\mathrm{M_Q}))$ is a morphism in $\mathbf{HDA}_L$ from Lemma 7. Here, $\mathrm{g} = \langle g, 1_L \rangle : \mathcal{T}_{\mathrm{p,X}}(\mathrm{M_P}) \to \mathcal{T}_{\varphi_q \circ m, \mathcal{T}_{q,Y}(M_Q)}(\mathrm{M_P})$ is defined as follows: $g(x,t) = (x,t)$, for all $x \in (M_P)_n$ and $t \in \overset{\circ}{\square}_n$ $(n \geq 0)$. We shall show that the diagram commutes. We have $f(\hat{p}(x,t)) = f(p(x)(t)) =$

$\check{f}(p(x))(t) = q(m(x))(t) = \hat{q}(m(x),t) = \hat{q}((m(x),\cdot)(t)) = \hat{q}(\varphi_q(m(x))(t)) = \hat{q}(\widehat{\varphi_q \circ m}(x,t)) = \hat{q}(\widehat{\varphi_q \circ m}(g(x,t)))$, for all point (x,t) in $\mathcal{T}_{\rm p,X}(\rm M_P)$. In order to use $\mathbf{TcP}_{\leq,L}$-openness of the morphism f, we have first to prove that g is a morphism in $\mathbf{THDA}_{\leq,L}$. Clearly, it is sufficient to show that $\|d_{(x,t)}g(\cdot)\|^2_{g((x,t))} \leq \|\cdot\|^1_{(x,t)}$, where $\|\cdot\|^1$ and $\|\cdot\|^2$ are the norms of the THDA $\mathcal{T}_{\rm p,X}(\rm M_P)$ and $\mathcal{T}_{\varphi_{\rm q}\circ \rm m,\mathcal{T}_{\rm q,Y}(\rm M_Q)}(\rm M_P)$, respectively. Due to the definition of the norms of the THDA $\mathcal{T}_{\varphi_{\rm q}\circ \rm m,\mathcal{T}_{\rm q,Y}(\rm M_Q)}(\rm M_P)$, $\mathcal{T}_{\rm q,Y}(\rm M_Q)$ and $\mathcal{T}_{\rm p,X}(\rm M_P)$, we have $\|d_{(x,t)}g(\cdot)\|^2_{g(x,t)} = \|d_{(x,t)}(\widehat{\varphi_q \circ m} \circ g)(\cdot)\|_{\widehat{\varphi_q \circ m}(g(x,t))} = \|d_{(x,t)}(\hat{q} \circ \widehat{\varphi_q \circ m} \circ g)(\cdot)\|_{\hat{q}(\widehat{\varphi_q \circ m}(g(x,t)))} = \|d_{(x,t)}(f \circ \hat{p})(\cdot)\|_{f(\hat{p}(x,t))} \leq \|d_{(x,t)}\hat{p}(\cdot)\|_{\hat{p}(x,t)} = \|\cdot\|^1_{(x,t)}$, because the diagram commutes and f is a morphism in $\mathbf{THDA}_{\leq,L}$. By Lemma 9, $\mathcal{T}_{\rm p,X}(\rm M_P)$ and $\mathcal{T}_{\rm q,Y}(\rm M_Q)$ are objects in $\mathbf{TcP}_{\leq,L}$. Since f is a $\mathbf{TcP}_{\leq,L}$-open morphism, there exists a morphism $\rm r : \mathcal{T}_{\rm q,Y}(\rm M_Q) \to X$ such that $\mathcal{T}_{\rm p,X}(\rm p) = \rm r \circ \mathcal{T}_{\varphi_{\rm q}\circ \rm m,\mathcal{T}_{\rm q,Y}(\rm M_Q)}(\varphi_{\rm q} \circ \rm m) \circ g$ and $\mathcal{T}_{\rm q,Y}(\rm q) = \rm f \circ \rm r$. Then, by virtue of Proposition 1 and Lemma 7, there exists a morphism $\mathcal{F}t(\rm r) \circ \varphi_{\rm q} : \rm M_Q \to \mathcal{F}t(\mathcal{T}_{\rm q,Y}(\rm M_Q)) \to \mathcal{F}t(X)$ in $\mathbf{HDA}_L$ such that $\rm p = \mathcal{F}t(\mathcal{T}_{\rm p,X}(\rm p)) \circ \varphi_{\rm p} = \mathcal{F}t(\rm r) \circ \mathcal{F}t(\mathcal{T}_{\varphi_{\rm q}\circ \rm m,\mathcal{T}_{\rm q,Y}(\rm M_Q)}(\varphi_{\rm q} \circ \rm m)) \circ \mathcal{F}t(\rm g) \circ \varphi_{\rm p} = \mathcal{F}t(\rm r) \circ \mathcal{F}t(\mathcal{T}_{\varphi_{\rm q}\circ \rm m,\mathcal{T}_{\rm q,Y}(\rm M_Q)}(\varphi_{\rm q} \circ \rm m)) \circ \varphi_{\varphi_{\rm q}\circ \rm m} = \mathcal{F}t(\rm r) \circ \varphi_{\rm q} \circ \rm m$, because $\mathcal{F}t(\rm g) : \mathcal{F}t(\mathcal{T}_{\rm p,X}(\rm M_P)) \to \mathcal{F}t(\mathcal{T}_{\varphi_{\rm q}\circ \rm m,\mathcal{T}_{\rm q,Y}(\rm M_Q)}(\rm M_P)$ is the identical morphism in $\mathbf{HDA}_L$. Analogously, we get $\rm q = \mathcal{F}t(\rm f) \circ \mathcal{F}t(\rm r) \circ \varphi_{\rm q}$. □

Further, we provide a behavioural criterion of $\mathbf{TcP}_{\leq,L}$-open morphisms which is crucial to formulate an open maps based characterization of timed hhp-bisimulation.

Theorem 3. *A morphism* $\rm f = \langle f, 1_L \rangle : X \to Y$ *in* $\mathbf{THDA}_{\leq,L}$ *is* $\mathbf{TcP}_{\leq,L}$*-open iff for all* $P \in \mathcal{CP}(\rm X)$ *the following holds:*

1. *if* $f(P) \xrightarrow{\delta_i^l} Q'$ *in* Y, *then* $P \xrightarrow{\delta_i^l} P'$ *and* $f(P') = Q'$ *for some* $P' \in \mathcal{CP}(\rm X)$,
2. *if* $f(P) \overset{(s,u,v)}{\longleftrightarrow} Q'$ *in* Y, *then* $P \overset{(s,u,v)}{\longleftrightarrow} P'$ *and* $f(P') = Q'$ *for some* $P' \in \mathcal{CP}(\rm X)$,
3. $d_u f$ *is an isometry for all reachable points* $u \in X$.

Proof. (⇒) Assume $\rm f = \langle f, 1_L \rangle : X \to Y$ to be a $\mathbf{TcP}_{\leq,L}$-open morphism. Consider the morphism $\mathcal{F}t(\rm f) = \langle \check{f}, 1_L \rangle : \mathcal{F}t(X) \to \mathcal{F}t(Y)$ in $\mathbf{HDA}_L$. Due to Proposition 2, $\mathcal{F}t(\rm f)$ is a $\mathbf{cP}_L$-open morphism. Then, by Theorem 1, items 1 and 2 hold. It remains to prove item 3.

Notice, for any reachable point $u \in X$, we get $u = r(v)$, for some point v in $\mathcal{T}_{\rm q,Y}(\rm M_Q)$ and some morphism $\rm r = \langle r, 1_L \rangle : \mathcal{T}_{\rm q,Y}(\rm M_Q) \to X$ from the diagram shown on the right side of Figure 9. Then, for every $\dot{u} \in T_uX$ there exists $\dot{v} \in T_vX_Q$ (X_Q is a topological space of $\mathcal{T}_{\rm q,Y}(\rm M_Q)$) such that $\dot{u} = d_vr(\dot{v})$. Hence, it holds that $\|\dot{u}\|_u = \|d_vr(\dot{v})\|_{r(v)} \geq \|d_{r(v)}f(d_vr(\dot{v}))\|_{f(r(v))} = \|d_v\hat{q}(\dot{v})\|_{\hat{q}(v)} = \|\dot{v}\|_v \geq \|d_vr(\dot{v})\|_{r(v)} = \|\dot{u}\|_u$. Therefore, d_uf is an isometry.

(⇐) Suppose that $\rm f = \langle f, 1_L \rangle : X \to Y$ is a morphism in $\mathbf{THDA}_{\leq,L}$. Also, assume that $\rm m = \langle m, 1_L \rangle : X_P \to X_Q$ is a morphism in $\mathbf{TcP}_{\leq,L}$ and $\rm p = \langle p, 1_L \rangle : X_P \to X$, $\rm q = \langle q, 1_L \rangle : X_Q \to Y$ are morphisms in $\mathbf{THDA}_{\leq,L}$ such that q ∘ m = f ∘ p. Then, using Proposition 1, we obtain the following commuting diagram in $\mathbf{HDA}_L$

$$\begin{array}{ccc} \mathcal{F}t(\mathrm{X_P}) & \xrightarrow{\mathcal{F}t(\mathrm{p})} & \mathcal{F}t(\mathrm{X}) \\ {\scriptstyle \mathcal{F}t(\mathrm{m})}\downarrow & & \downarrow{\scriptstyle \mathcal{F}t(\mathrm{f})} \\ \mathcal{F}t(\mathrm{X_Q}) & \xrightarrow[\mathcal{F}t(\mathrm{q})]{} & \mathcal{F}t(\mathrm{Y}) \end{array}$$

Moreover, due to Lemma 10, $\mathcal{F}t(\mathrm{X_P})$ and $\mathcal{F}t(\mathrm{X_Q})$ are objects in $\mathbf{cP}_L$, and hence, $\mathcal{F}t(\mathrm{m})$ is a morphism in $\mathbf{cP}_L$. Since $\mathcal{F}t(\mathrm{f})$ meets the conditions of Theorem 1, we can find a morphism $\mathrm{r}' = \langle r', 1_L\rangle : \mathcal{F}t(\mathrm{X_Q}) \to \mathcal{F}t(\mathrm{X})$ in $\mathbf{HDA}_L$ such that $\mathcal{F}t(\mathrm{p}) = \mathrm{r}' \circ \mathcal{F}t(\mathrm{m})$ and $\mathcal{F}t(\mathrm{q}) = \mathcal{F}t(\mathrm{f}) \circ \mathrm{r}'$. Define a mapping $r : X_Q \to X$ as follows: $r(x(t)) = r'(x)(t)$, for all $x \in (X_Q)_n$ and $t \in \Box_n$ $(n \geq 0)$. Clearly, r is a well-defined mapping. Since X_Q is a $\Box$-topological space and r satisfies condition 2a) of Definition 7, r is a continuous mapping, by Lemma 5. Moreover, we have $p(x(t)) = \check{p}(x)(t) = r'(\check{m}(x))(t) = r(\check{m}(x)(t)) = r(m(x(t)))$ for all $x(t) \in X_P$, i.e. $p = r \circ m$. Similarly, we get $q = f \circ r$. Then, due to item 3, it holds that $\|d_v r(\dot{v})\|_{r(v)} = \|d_{r(v)} f(d_v r(\dot{v}))\|_{f(r(v))} = \|d_v q(\dot{v})\|_{q(v)} \leq \|\dot{v}\|_v$, for all $v \in X_Q$ and $\dot{v} \in T_v X_Q$. Thus, it is obvious that $\mathrm{r} = \langle r, 1_L\rangle$ is a morphism in $\mathbf{THDA}_{\leq,L}$ satisfying the following equations: p = r ∘ m and q = f ∘ r. This means that f is a $\mathbf{TcP}_{\leq,L}$-open morphism in $\mathbf{THDA}_{\leq,L}$. □

Finally, the coincidence of $\mathbf{TcP}_{\leq,L}$-bisimulation and timed hhp-bisimulation is established.

Theorem 4. *Two timed HDA (with the same set L of actions) are* $\mathbf{TcP}_{\leq,L}$-*bisimilar iff they are thhp-bisimilar.*

Proof. (⇒) Suppose a span $\mathrm{X} \xleftarrow{\mathrm{f_X}} \mathrm{Z} \xrightarrow{\mathrm{f_Y}} \mathrm{Y}$ of $\mathbf{TcP}_{\leq,L}$-open morphisms $\mathrm{f}_X = \langle f_X, 1_L\rangle$ and $\mathrm{f}_Y = \langle f_Y, 1_L\rangle$ in $\mathbf{THDA}_{\leq,L}$. We shall prove that X and Y are thhp-bisimilar. Construct a relation $\mathcal{R} = \{(f_X(P), f_Y(P)) \mid P \in \mathcal{CP}(\mathrm{Z})\}$. Take an arbitrary $P = p_0 \ldots p_k \in \mathcal{CP}(\mathrm{Z})$. Since f_X and f_Y are $\mathbf{TcP}_{\leq,L}$-open morphisms, the cubical paths $f_X(P)$ and $f_Y(P)$ are d-related, or, in detail, $\|d_{p_i(t)} f_X(d_t p_i(\dot{t}))\|_{f_X(p_i(t))} = \|d_t p_i(\dot{t})\|_{p_i(t)} = \|d_{p_i(t)} f_Y(d_t p_i(\dot{t}))\|_{f_Y(p_i(t))}$, as each point $p_i(t) \in p_i(\mathring{\Box}_{\dim p_i})$ is reachable for all $t \in \mathring{\Box}_{\dim p_i}$ and $0 \leq i \leq k$. On the other hand, due to Proposition 2, morphisms $\mathcal{F}t(\mathrm{f}_X) = \langle \check{f}_X, 1_L\rangle$ and $\mathcal{F}t(\mathrm{f}_Y) = \langle \check{f}_Y, 1_L\rangle$ are $\mathbf{cP}_L$-open. From the reasonings in the proof of Theorem 2, it follows that the relation $\widetilde{\mathcal{R}} = \{(\check{f}_X(Q),\ \check{f}_Y(Q)) \mid Q \in \mathcal{CP}(\mathcal{F}t(\mathrm{Z}))\}$ is hhp-bisimulation between $\mathcal{F}t(\mathrm{X})$ and $\mathcal{F}t(\mathrm{Y})$. It is easy to see that $\widetilde{\mathcal{R}} = \mathcal{R}_{\mathcal{F}t}$. Clearly, $(i_0^{\mathrm{X}}, i_0^{\mathrm{Y}}) \in \mathcal{R}$. Thus, X and Y are thhp-bisimular.

(⇐) Assume $\mathcal{R}$ to be a thhp-bisimulation between THDA X and Y (with the same set L of actions). We have to construct a span $\mathrm{X} \xleftarrow{\mathrm{f_X}} \mathrm{Z} \xrightarrow{\mathrm{f_Y}} \mathrm{Y}$ of $\mathbf{TcP}_{\leq,L}$-open morphisms $\mathrm{f_X} = \langle f_X, 1_L\rangle$ and $\mathrm{f_Y} = \langle f_Y, 1_L\rangle$ in $\mathbf{THDA}_{\leq,L}$.

By Definition 8, $\mathcal{R}_{\mathcal{F}t}$ is an hhp-bisimulation between $\mathcal{F}t(\mathrm{X})$ and $\mathcal{F}t(\mathrm{Y})$. Due to the reasonings in the proof of Theorem 2, we can find a span $\mathcal{F}t(\mathrm{X}) \xleftarrow{\mathrm{pr}_1} \langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle \xrightarrow{\mathrm{pr}_2} \mathcal{F}t(\mathrm{Y})$ of $\mathbf{cP}_L$-open morphisms $\mathrm{pr}_1 = \langle pr_1, 1_L\rangle$ and $\mathrm{pr}_2 = \langle pr_2, 1_L\rangle$ in $\mathbf{HDA}_L$. The mappings $\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\mathrm{pr}_1) : \mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle) \to \mathrm{X}$ and $\mathcal{T}_{\mathrm{pr}_2,\mathrm{Y}}(\mathrm{pr}_2) : \mathcal{T}_{\mathrm{pr}_2,\mathrm{Y}}(\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle) \to \mathrm{Y}$ are morphisms in $\mathbf{THDA}_{\leq,L}$ by Lemma 6. To construct the required span we need to show that

$$\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle) = \mathcal{T}_{\mathrm{pr}_2,\mathrm{Y}}(\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle).$$

It is sufficient to prove the coincidence of the norms $\|\cdot\|^1$ and $\|\cdot\|^2$ of the THDA $\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle)$ and $\mathcal{T}_{\mathrm{pr}_2,\mathrm{Y}}(\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle)$, respectively. Let Z be the common topological space of $\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle)$ and $\mathcal{T}_{\mathrm{pr}_2,\mathrm{Y}}(\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle)$. Then, for all $w = (\langle P_{\mathcal{F}t}, Q_{\mathcal{F}t}\rangle, t) \in Z$ with $P \in \mathcal{CP}_{p_k}(\mathrm{X})$, $Q \in \mathcal{CP}_{q_k}(\mathrm{Y})$, $(P,Q) \in \mathcal{R}$, $t \in \mathring{\square}_n$, and $\dot{t} \in T_w Z$, we have $\|\dot{t}\|^1_w = \|d_w \widehat{pr}_1(\dot{t})\|_{\widehat{pr}_1(w)} = \|d_t p_k(\dot{t})\|_{p_k(t)} = \|d_t q_k(\dot{t})\|_{q_k(t)} = \|d_w \widehat{pr}_2(\dot{t})\|_{\widehat{pr}_2(w)} = \|\dot{t}\|^2_w$.

It remains to show that the morphism $\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\mathrm{pr}_1)$ is $\mathbf{TcP}_{\leq,L}$-open, due to Theorem 3 (the proof of $\mathbf{TcP}_{\leq,L}$-openness of the morphism $\mathcal{T}_{\mathrm{pr}_2,\mathrm{Y}}(\mathrm{pr}_2)$ is similar). By Lemma 7, we have $\mathcal{F}t(\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\mathrm{pr}_1)) = \mathrm{pr}_1 \circ \psi_{\mathrm{pr}_1}$. Clearly, ψ_{pr_1} is a $\mathbf{cP}_L$-open morphism. Since the composition of $\mathbf{cP}_L$-open morphisms in $\mathbf{HDA}_L$ is a $\mathbf{cP}_L$-open morphism in $\mathbf{HDA}_L$, $\mathcal{F}t(\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\mathrm{pr}_1))$ is a $\mathbf{cP}_L$-open morphism in $\mathbf{HDA}_L$. Hence, using Theorem 1 for $\mathcal{F}t(\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\mathrm{pr}_1))$, items 1 and 2 of Theorem 3 hold for $\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\mathrm{pr}_1)$. Due to the definition of the norm on $\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\rangle)$, item 3 of Theorem 3 holds as well. □

4 Conclusion

The paper focuses on open maps characterizations of hhp-bisimulation on HDA and timed hhp-bisimulation on THDA. We remark that the equivalences have been attacked using homotopy techniques, following the papers [7, 23]. In particular, guided by our intuitive understanding of what it means for a higher dimensional automata model to be simulated by another one, we have defined categories of HDA and THDA and accompanying (sub)categories of observations, to which the corresponding notions of open maps have been developed. We have used the open maps framework [15] to obtain abstract bisimulations which have been established to coincide with the mentioned above bisimulations on HDA and THDA. The open maps based bisimilarity makes possible a uniform definition of bisimulation over different models presented as categories and allows one to apply general results from the categorical setting (e.g. the existence of canonical models and characteristic games and logics) to concrete behavioural equivalences. Notice, all the results of the paper are valid for the category $\mathbf{THDA}_\star$, where $\star \in \{\cdot, \leq, =\}$[12].

As a matter of future work, it would be interesting to extend the results obtained in the paper [5] to weak variant of bisimulation on HDA and THDA,

[12] THDA and morphisms from Definition 7, whose first component is an isometry, constitute a category $\mathbf{THDA}_=$.

combining open maps and presheaf approaches. Also, we plan some investigation on coalgebraic characterizations [22] of bisimulation in the setting of HDA and THDA.

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