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SOME FORMAL ASPECTS OF SUBDEFINITE MODELS

Preprint

49

Novosibirsk 1998

General methods for problem solving are objects of study in artificial intelligence. This work describes *subdefinite models* as a framework for solving *constraint satisfaction problems*. Such problems arise in many fields of science and engineering. The use of the method of constraint propagation in subdefinite models makes it possible to estimate the set of all solutions of such a problem.

Foundations of subdefinite models are given in the paper. The notion of subdefinite extension of a many-sorted model is proposed, and the types of such extensions are discussed. Both denotational and operational semantics of constraint propagation in subdefined models are defined and their equivalence is proved.

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**НЕКОТОРЫЕ ФОРМАЛЬНЫЕ АСПЕКТЫ
НЕДООПРЕДЕЛЕННЫХ МОДЕЛЕЙ**

Препринт
49

Новосибирск 1998

Универсальные методы решения задач изучаются в рамках научной дисциплины “искусственный интеллект”. Настоящая работа посвящена описанию *недоопределенных моделей* как аппарата для решения *задач удовлетворения ограничений*. Такие задачи возникают во многих областях науки и техники. Использование метода распространения ограничений в недоопределенных моделях позволяет оценивать множество всех решений такой задачи.

В статье даются обоснования недоопределенных моделей. Предлагается понятие недоопределенного расширения многосортной модели, и обсуждаются различные типы таких расширений. Определяются денотационная и операционная семантики распространения ограничений в недоопределенных моделях и доказывается их эквивалентность.

1. INTRODUCTION

The constraint programming (CP) paradigm is one of the most fruitful and commonly used in the present-day computer science [1–3]. This paper is devoted to the description of *subdefinite models* apparatus as one of the varieties of the CP approach.

Subdefiniteness allows us to use fruitfully all reliable data concerning real-world objects, including uncertain, noisy, incomplete and imprecise data. The notion of subdefiniteness was proposed by A.S. Narin'yani at the beginning of the 1980s [4]. He has developed a formal apparatus which extends the set theory and allows the representation of partially known (subdefinite) sets as well as automatic solution of set-theoretic problems incorporating subdefinite sets. This idea has been extended to a formal apparatus of active data types, which allows us to build subdefinite extensions of a wider class of objects, concepts, and events [5].

Given *variables*, their *domains*, and *constraint formulae*, one can define the *Constraint Satisfaction Problem* (CSP) as a problem of finding values of all variables in their domains (a *solution*) satisfying the constraint formulae. Obviously, there exist algorithms to find solutions of a very small part of such problems even for well known domains and constraints, and only a few existing algorithms are effective. Finding of all solutions is a more difficult task. Often one would like to get some estimation of a set of all solutions of a given problem before starting the search for a solution (to reduce the domains of variables) using some algorithm (in reduced domains). The reduction of domains can be done by means of a *constraint propagation algorithm*. The algorithm for finite domains and binary constraints has been initially proposed in [1] and then has been generalized for integers [2,3], reals [6,7] and mixed (finite, integer, real) computation domains [8–10].

This work describes the apparatus of *subdefinite models* and their application to constraint satisfaction in arbitrary domains. We regard the CSP as a first-order formula of a given many-sorted signature whose variables have existential quantification and all atoms are joined with conjunctions. Solutions of this CSP are witnesses of a realization of this formula in the given many-sorted model of the same signature. The notion of a *subdefinite extension* of a domain, functions and predicates over it allows us to regard a constraint propagation algorithm as a variety of methods of successive approximation. Its denotational semantics can be expressed in terms of the greatest fixed point of some mappings in subdefinite extensions of domains. Operational semantics is defined in terms of an abstract machine with states

and transition rules. The correctness and the equivalence of these semantics are proved. Several types of subdefinite extensions are proposed and their properties are explored.

The work is organized as follows. In Section 2, the basic notions of many-sorted models are presented. A subdefinite extension of the models is described in Section 3. Denotational semantics of constraint propagation in subdefinite models (in terms of the greatest fixed point of the family of some mappings) is defined in Section 4. Section 5 is devoted to the definition of operational semantics of constraint propagation. In Section 6, we examine the types of such subdefinite extensions. Finally, Section 7 concludes the paper.

2. BASIC NOTIONS

A traditional problem of subdefinite models is to find all solutions of an existential conjunction in a sorted model. These models completely correspond to models from [11]. To describe denotational semantics of this problem, we should specify a subdefinite extension of the corresponding sorted model. We briefly redefine basic definitions of sorted models.

Definition 1. A *many-sorted signature* is a triple $\Sigma = (S, F, P)$ where

- S is a set of *sorts* (elements of S are names of different domains); we denote the set of all chains of elements of S (including an empty chain, λ) by S^* , i. e. $S^* = \{\lambda\} \cup S \cup S^2 \cup \dots$, we also define $S^+ = S^* \setminus \{\lambda\}$;
- F is an $(S^* \times S)$ -indexed family of sets of *operators (function symbols)* (i. e. $F = \{F_{w,s} \mid w \in S^*, s \in S\}$); $F_{\lambda,s}$ is called a set of *constants* of the sort s ;
- $P = \{P_w \mid w \in S^+\}$ is a family of *predicate symbols* containing the predicate symbol of equality, $= \in P_{ss}$, for each sort $s \in S$.

Example 1. Consider the signature $\Sigma^1 = (S^1, F^1, P^1)$. Let $real \in S^1$, $\{+, *\} \subseteq F_{real\ real}^1$, $\{=, \leq\} \subseteq P_{real\ real}^1$.

Definition 2. For a many-sorted signature $\Sigma = (S, F, P)$, a *many-sorted Σ -model* M is a *first-order structure* consisting of:

- a family of *carriers* s^M for all $s \in S$ (for $w = s_1 \dots s_n$ we denote by w^M the Cartesian product $s_1^M \times \dots \times s_n^M$),
- a family of *functions* $f^M : w^M \rightarrow s^M$ for all $f \in F_{w,s}$, if $w = \lambda$, then $f^M \in s^M$,
- a family of *predicates* $p^M \subseteq w^M$ for all $p \in P_w$, the predicate of equality $=^M \subseteq (ss)^M$ is $\{(a, a) \mid a \in s^M\}$ for all $s \in S$.

Example 2. Consider Σ^1 -model \mathcal{R} (the many-sorted signature Σ^1 was defined in Example 1), where $real^{\mathcal{R}}$ is the set of all real numbers, $+\mathcal{R}$ is the operation of real addition, $*^{\mathcal{R}}$ is the real multiplication, $=^{\mathcal{R}}$ is the equality, and $\leq^{\mathcal{R}}$ is the relation "less or equal than" between two real numbers.

Definition 3. Let $\Sigma = (S, F, P)$ be a many-sorted signature and X be an S -sorted set (of *variables*) such that $X_{s'} \cap X_{s''} = \emptyset$ for $s' \neq s''$, and $X_s \cap F_{\lambda, s} = \emptyset$ for any $s \in S$. We define $\Sigma(X)$ -terms as the smallest S -indexed set $T_{\Sigma}(X)$, such that

- X_s and $F_{\lambda, s} \subseteq T_{\Sigma}(X)_s$ for all $s \in S$;
- if $f \in F_{w, s}$ and $t_i \in T_{\Sigma}(X)_{s_i}$ for $w = s_1 \dots s_n \in S^+$, then the string $f(t_1, \dots, t_n)$ belongs to $T_{\Sigma}(X)_s$.

We define $\Sigma(X)$ -atoms as expressions of the form $p(t_1, \dots, t_n)$, where $t_i \in T_{\Sigma}(X)_{s_i}$, $p \in P_w$, $w = s_1 \dots s_n \in S^+$.

Definition 4. Let $\Sigma = (S, F, P)$ be a many-sorted signature, X be an S -sorted set of variables, M be a Σ -model. Any S -sorted function* $v : X \rightarrow M$ is called an *estimate* (or *valuation*) of variables X in Σ -model M . An *extension* of the estimate v to the set $T_{\Sigma}(X)$ is a function $v^* : T_{\Sigma}(X) \rightarrow M$ defined as follows. For $t \in T_{\Sigma}(X)_s$, $s \in S$:

$$v_s^*(t) = \begin{cases} v_s(x), & \text{if } t \equiv x \text{ for any } x \in X_s, \\ c^M, & \text{if } t \equiv c \text{ for any } c \in F_{\lambda, s}, \\ f^M(v_{s_1}^*(t_1), \dots, v_{s_n}^*(t_n)), & \text{if } t \equiv f(t_1, \dots, t_n) \text{ for } f \in F_{w, s}, \\ & w = s_1 \dots s_n \in S^+, t_i \in T_{\Sigma}(X)_{s_i}. \end{cases}$$

Definition 5. Given a many-sorted signature $\Sigma = (S, F, P)$ and an S -sorted set of variables X , we define a $\Sigma(X)$ -*existential conjunction* (of atoms) to be an existential formula of the form

$$(\exists X) \quad A_1, \dots, A_n$$

where A_1, \dots, A_n are $\Sigma(X)$ -atoms. We say that a Σ -model M *realizes* this conjunction (or that this conjunction *is realized* in M), if there exists an estimate $v : X \rightarrow M$ (called a *witness*), such that for each atom $p(t_1, \dots, t_k) \in \{A_1, \dots, A_n\}$

$$(v_{s_1}^*(t_1), \dots, v_{s_k}^*(t_k)) \in p^M$$

*An S -sorted function $v : X \rightarrow M$ is actually a family of mappings

$$v = \{v_s : X_s \rightarrow s^M \mid s \in S\}.$$

where $p \in P_w$, $w = s_1 \dots s_k \in S^+$, and $t_i \in T_\Sigma(X)_{s_i}$, $i = 1, \dots, k$.

We will only deal with conjunctions in a special form called *constraint satisfaction problem*.

Definition 6. Given a many-sorted signature $\Sigma = (S, F, P)$ and an S -sorted set of variables X , we define $\Sigma(X)$ -*constraint* as a $\Sigma(X)$ -atom having one of the following forms:

- $p(x_1, \dots, x_n)$, where $p \in P_w$, $w = s_1 \dots s_n \in S^+$, $x_i \in X_{s_i}$ for $i = 1, \dots, n$, $n > 0$, and all the variables x_1, \dots, x_n are different;
- $f(x_1, \dots, x_n) = y$, where $f \in F_{w,s}$, $w = s_1 \dots s_n \in S^*$, $s \in S$, $y \in X_s$, $x_i \in X_{s_i}$ for $i = 1, \dots, n$, $n \geq 0$, and all the variables x_1, \dots, x_n, y are different.

A $\Sigma(X)$ -*constraint satisfaction problem*, or *CSP* is a $\Sigma(X)$ -existential conjunction, all its atoms are constraints.

Proposition 1. Any $\Sigma(X)$ -existential conjunction

$$(\exists X) \quad A_1, \dots, A_m \tag{1}$$

can be transformed in finitely many steps into an equivalent $\Sigma(X \cup Y)$ -constraint satisfaction problem[†]

$$(\exists X \cup Y) \quad B_1, \dots, B_{m+k} \tag{2}$$

where Y is an S -sorted set of variables which does not intersect X . The stated equivalence has the following meaning: for any Σ -model M and a witness $v : X \rightarrow M$ of the realization of (1) in M , there exists a witness $v' : X \cup Y \rightarrow M$ of the realization of (2) in M , such that $v'|_X = v$ and, for any witness $u : X \cup Y \rightarrow M$ of the realization of (2) in M , the S -sorted function $u|_X$ is a witness of the realization of (1).

Before proving the proposition, let us proceed with an example.

Example 3. Let X be an S^1 -sorted set of variables (S^1 as defined in Example 1), where $x, y, z, w \in X_{real}$. Consider the following $\Sigma^1(X)$ -conjunction:

$$(\exists X) \quad x * y + z < w.$$

The equivalent $\Sigma^1(X \cup Y)$ -constraint satisfaction problem is:

$$(\exists X \cup Y) \quad x * y = t_1, \quad t_1 + z = t_2, \quad t_2 < w,$$

[†]The union of two S -sorted sets A and B is defined as follows:

$$A \cup B = \{A_s \cup B_s \mid s \in S\}.$$

where $t_1, t_2 \in Y_{real}$. It is equivalent to the first one in the sense of Proposition 1.

Proof. The meaning of the transformation from (1) to (2) is the recursive reduction of terms. If (1) has an atom $p(t_1, \dots, t_n)$, then (2) has an atom $p(y_1, \dots, y_n)$ and all the atoms from the sets $Eq(y_i, t_i)$ ($i = 1, \dots, n$), where $Eq(y, t)$ can be recursively defined as

$$\begin{aligned} Eq(y, x) &= \{y = x\} \text{ for all } s \in S, x \in X_s, y \in Y_s, \\ Eq(y, c) &= \{y = c\} \text{ for all } c \in F_{\lambda, s}, s \in S, y \in Y_s. \\ Eq(y, f(t_1, \dots, t_n)) &= \{Eq(y_1, t_1), \dots, Eq(y_n, t_n), f(y_1, \dots, y_n) = y\} \\ &\text{ for all } f \in F_{s_1 \dots s_n, s}, s_1, \dots, s_n, s \in S, y \in Y_s, \\ &\text{ where } y_i \in Y_{s_i}. \end{aligned}$$

We assume that each variable from Y has a single occurrence in the atoms of (2). This recursive transformation can be performed in finitely many steps and its result is a $\Sigma(X \cup Y)$ -existential conjunction (2). Consider an arbitrary Σ -model M . Let $v : X \rightarrow M$ be a witness of (1) in M , and $v^* : T_\Sigma(X) \rightarrow M$ be its extension to the set of $\Sigma(X)$ -terms. If we have added the atoms from $Eq(y, t)$ for $y \in Y_s, t \in T_\Sigma(X)_s$ to (2), then we define $v'_s(y) = v^*_s(t)$. For any $s \in S, x \in X_s$, we define $v'_s(x) = v_s(x)$. It is easy to see that v' is a witness of (2). Let now $u : X \cup Y \rightarrow M$ be a witness of (2). We can reverse our transformation (i. e. perform a transformation from (2) to (1)) and see that $u|_X$ will be a witness of the realization of (1).

3. SUBDEFINITE EXTENSIONS

An estimate (see Definition 2) evaluates each variable with a single value from the universe of its sort. A fruitful idea is the valuation of variables with sets of values rather than single values. The theoretical framework which allows such valuation is introduced in this section.

Definition 7. Let $\Sigma = (S, F, P)$ be a many-sorted signature, and M be a Σ -model. A *subdefinite extension* (*SD-extension*) of a Σ -model M is a Σ -model $*M$ defined as follows:

- every carrier $s^{*M}, s \in S$, satisfies the following conditions:
 - s^{*M} is a finite set of subsets of s^M ;
 - $\emptyset \in s^{*M}$;
 - $s^M \in s^{*M}$;
 - if $\alpha_1 \in s^{*M}$ and $\alpha_2 \in s^{*M}$, then $\alpha_1 \cap \alpha_2 \in s^{*M}$.

The elements of s^M will be called *subdefinite values*, or *SD-values*. Let $w = s_1 \dots s_n \in S^+$. We will also consider each element

$$\alpha = (\alpha_1, \dots, \alpha_n) \in w^{*M}$$

(remember that $w^{*M} = s_1^{*M} \times \dots \times s_n^{*M}$) as the set

$$\alpha = \alpha_1 \times \dots \times \alpha_n \subseteq w^M.$$

The properties of subdefinite extension guarantee *unique representation*, or *approximation* of any subset $\xi \subseteq w^M$ (which will be denoted by $w^{*M}[\xi] \in w^{*M}$) in the SD-extension *M , namely:

$$w^{*M}[\xi] = \bigcap_{\xi \subseteq \alpha \in w^{*M}} \alpha. \quad (3)$$

Thus, the representation of the set ξ in w^{*M} is the minimal element of the w^{*M} which contains ξ .

- the operations of the model *M are defined as follows. For $f \in F_{w,s}$, where $w = s_1 \dots s_n \in S^*$, $s \in S$

$$f^{*M}(\alpha_1, \dots, \alpha_n) = s^{*M}[\{f^M(a_1, \dots, a_n) \mid a_1 \in \alpha_1, \dots, a_n \in \alpha_n\}]. \quad (4)$$

Note that f^{*M} is a total function although f^M can be partial.

- the predicates of the model *M are defined as follows. For $p \in P_w$, where $w = s_1 \dots s_n \in S^+$, and $\alpha \in w^{*M}$

$$\alpha \in p^{*M} \text{ iff } \alpha = w^{*M}[\alpha \cap p^M]. \quad (5)$$

We also define predicates $f^{*M} \subseteq w^{*M} \times s^{*M}$ for each $f \in F_{w,s}$, where $w = s_1 \dots s_n \in S^+$, $s \in S$ as follows. Let $f^M \subseteq w^M \times s^M$ be the graph of the function $f^M : w^M \rightarrow s^M$:

$$f^M = \{(a_1, \dots, a_n, a_{n+1}) \in w^M \times s^M \mid f^M(a_1, \dots, a_n) = a_{n+1}\}.$$

Then for $\alpha \in w^{*M} \times s^{*M}$

$$\alpha \in f^{*M} \text{ iff } \alpha = (ws)^{*M}[\alpha \cap f^M].$$

From now, we will regard a $\Sigma(X)$ -constraint $f(x_1, \dots, x_n) = y$ as a constraint $f(x_1, \dots, x_n, y)$, i. e. we will use f as a predicate symbol.

Note that the definition of the predicate of equality in an SD-extension of a model is correct, i. e. the following proposition holds.

Proposition 2. *Let $\Sigma = (S, F, P)$ be a many-sorted signature, *M be an SD-extension of a Σ -model M , $\alpha_1, \alpha_2 \in s^{*M}$, $s \in S$. Then $\alpha_1 = {}^*M \alpha_2$ holds iff α_1 and α_2 are the same set.*

Proof. Let $\alpha_1 = {}^*M \alpha_2$ holds. From (5) we have

$$\alpha_1 \subseteq s^{*M}[\{\{a_1 \in s^M \mid (\exists a_2 \in \alpha_2) \quad a_1 =^M a_2\}\}] = s^{*M}[\alpha_2] = \alpha_2$$

(the last is true since $\alpha_2 \in s^{*M}$), and

$$\alpha_2 \subseteq s^{*M}[\{\{a_2 \in s^M \mid (\exists a_1 \in \alpha_1) \quad a_1 =^M a_2\}\}] = s^{*M}[\alpha_1] = \alpha_1,$$

i. e. $\alpha_1 = \alpha_2$ as subsets of s^M . Conversely, let $\alpha_1 = \alpha_2$. Then

$$\alpha_1 \subseteq \alpha_2 = s^{*M}[\{\{a_1 \in s^M \mid (\exists a_2 \in \alpha_2) \quad a_1 =^M a_2\}\}],$$

and

$$\alpha_2 \subseteq \alpha_1 = s^{*M}[\{\{a_2 \in s^M \mid (\exists a_1 \in \alpha_1) \quad a_1 =^M a_2\}\}],$$

i. e. $\alpha_1 = {}^*M \alpha_2$ holds.

The reason for such definition of an SD-extension is the following. The functions and the predicates over sets are approximations of ones over single values. The result of an SD-extension of a function on some sets is defined as the approximation of the set of all results of a function obtained on combinations of single values from these sets. An SD-extension of a predicate holds on some sets iff these sets are *SD-consistent* w.r.t. this predicate in M . The notion of *consistency* was firstly proposed in [1] in the following sense (we rewrite the definition of consistency in our terms).

Definition 8. Let $\Sigma = (S, F, P)$ be a many-sorted signature, M be a Σ -model, and $p \in P_{s_1 \dots s_n}$ for $s_1, \dots, s_n \in S$. Any sets of values

$$\xi_1 \subseteq s_1^M, \dots, \xi_n \subseteq s_n^M$$

(of arguments of a predicate p^M) are *consistent* w.r.t. p^M iff for all $i = 1, \dots, n$ and $a_i \in \xi_i$ there exist

$$a_1 \in \xi_1, \dots, a_{i-1} \in \xi_{i-1}, a_{i+1} \in \xi_{i+1}, \dots, a_n \in \xi_n$$

such that $(a_1, \dots, a_n) \in p^M$.

We extend the notion of consistency to SD-consistency as follows.

Definition 9. Let $\Sigma = (S, F, P)$ be a many-sorted signature, *M be an SD-extension of a Σ -model M , and $p \in P_{s_1 \dots s_n}$ for $s_1, \dots, s_n \in S$. Any SD-values

$$\alpha_1 \in s_1^{*M}, \dots, \alpha_n \in s_n^{*M}$$

are *SD-consistent* w.r.t. p^M iff they are approximations of a consistent sets, i. e. there exist

$$\xi_1 \subseteq s_1^M, \dots, \xi_n \subseteq s_n^M$$

such that ξ_1, \dots, ξ_n are consistent w.r.t. p^M , and $s_i^{*M}[\xi_i] = \alpha_i$ for $i = 1, \dots, n$.

We want to prove that a predicate holds in an SD-extension of a model iff their arguments are SD-consistent. Let us begin with the following

Proposition 3. *Let $\Sigma = (S, F, P)$ be a many-sorted signature, M be a Σ -model, *M be its SD-extension, $w = s_1 \dots s_n \in S^+$, $\xi \subseteq w^M$. For $i = 1, \dots, n$ define $\pi_i(\xi)$ as *i -th projection* of ξ , i. e.*

$$\pi_i(\xi) = \{a_i \in s_i^M \mid (\forall k \neq i)(\exists a_k \in s_k^M) (a_1, \dots, a_n) \in \xi\}.$$

Then the following assertion holds:

$$w^{*M}[\xi] = s_1^{*M}[\pi_1(\xi)] \times \dots \times s_n^{*M}[\pi_n(\xi)].$$

Proof. Let $\alpha \in w^{*M}$. Remember that $\alpha = (\alpha_1, \dots, \alpha_n) = \alpha_1 \times \dots \times \alpha_n$ with $\alpha_i \in s_i^{*M}$ for $i = 1, \dots, n$. Let us prove that

$$\xi \subseteq \alpha \text{ iff } \pi_1(\xi) \times \dots \times \pi_n(\xi) \subseteq \alpha.$$

The second assertion is a consequence of the first one since the projection is a monotone mapping (w.r.t. relation \subseteq over sets) and $\pi_1(\alpha) \times \dots \times \pi_n(\alpha) = \alpha_1 \times \dots \times \alpha_n = \alpha$. Let the second assertion holds. Since

$$\xi \subseteq \pi_1(\xi) \times \dots \times \pi_n(\xi),$$

we have $\xi \subseteq \alpha$.

Consequently,

$$\begin{aligned} w^{*M}[\xi] &= \bigcap_{\xi \subseteq \alpha \in w^{*M}} \alpha = \bigcap_{\pi_1(\xi) \times \dots \times \pi_n(\xi) \subseteq \alpha_1 \times \dots \times \alpha_n \in w^{*M}} \alpha_1 \times \dots \times \alpha_n = \\ &= \left(\bigcap_{\pi_1(\xi) \subseteq \alpha_1 \in s_1^{*M}} \alpha_1 \right) \times \dots \times \left(\bigcap_{\pi_n(\xi) \subseteq \alpha_n \in s_n^{*M}} \alpha_n \right) = \end{aligned}$$

$$= s_1^{*M}[\pi_1(\xi)] \times \dots \times s_n^{*M}[\pi_n(\xi)].$$

The following proposition is a reformulation of an SD-extension of a predicate.

Proposition 4. *Let $\Sigma = (S, F, P)$ be a many-sorted signature, M be a Σ -model, *M be an SD-extension of M , $p \in P_w$ for $w = s_1 \dots s_n \in S^+$, $\alpha_i \in s_i^{*M}$, $i = 1, \dots, n$. The following assertions are equivalent:*

- $\alpha_1, \dots, \alpha_n$ are SD-consistent w.r.t. p^M ,
- $(\alpha_1, \dots, \alpha_n) \in p^{*M}$.

Proof. Let the first assertion hold. Then there exist $\xi_1 \subseteq s_1^M, \dots, \xi_n \subseteq s_n^M$ which are consistent w.r.t. p^M , i. e. for all $i = 1, \dots, n$, $a_i \in \xi_i$ and $k \neq i$ there exists $a_k \in \xi_k$ such that $(a_1, \dots, a_n) \in p^M$. We can rewrite this proposition as follows: for any $i = 1, \dots, n$

$$\xi_i = \pi_i(\xi_1 \times \dots \times \xi_n \cap p^M),$$

Since $\pi_i(\dots)$ and $s_i^{*M}[\dots]$ are monotone (w.r.t. relation \subseteq) mappings and $\xi_i \subseteq \alpha_i$ for $i = 1, \dots, n$, the following inclusions hold.

$$\begin{aligned} \alpha_i = s_i^{*M}[\xi_i] &= s_i^{*M}[\pi_i(\xi_1 \times \dots \times \xi_n \cap p^M)] \subseteq \\ &\subseteq s_i^{*M}[\pi_i(\alpha_1 \times \dots \times \alpha_n \cap p^M)] \subseteq \\ &\subseteq s_i^{*M}[\pi_i(\alpha_1 \times \dots \times \alpha_n)] = s_i^{*M}[\alpha_i] = \alpha_i, \end{aligned}$$

i. e. for $i = 1, \dots, n$

$$\alpha_i = s_i^{*M}[\pi_i(\alpha_1 \times \dots \times \alpha_n \cap p^M)].$$

We can rewrite it (see Proposition 3) as follows:

$$\alpha = w^{*M}[\alpha \cap p^M],$$

for $\alpha = (\alpha_1, \dots, \alpha_n) = \alpha_1 \times \dots \times \alpha_n$.

Let the second assertion hold. Define $\xi_1 \subseteq s_1^M, \dots, \xi_n \subseteq s_n^M$ as follows:

$$\xi_i = \pi_i(\alpha_1 \times \dots \times \alpha_n \cap p^M), \quad i = 1, \dots, n.$$

Obviously (see Proposition 3), $s_i^{*M}[\xi_i] = \alpha_i$ for $i = 1, \dots, n$. We have

$$\alpha_1 \times \dots \times \alpha_n \cap p^M \subseteq \xi_1 \times \dots \times \xi_n \subseteq \alpha_1 \times \dots \times \alpha_n,$$

therefore

$$\xi_1 \times \dots \times \xi_n \cap p^M = \alpha_1 \times \dots \times \alpha_n \cap p^M,$$

i. e. for $i = 1, \dots, n$

$$\xi_i = \pi_i(\xi_1 \times \dots \times \xi_n \cap p^M).$$

This means that

$$(\forall i = 1, \dots, n)(\forall a_i \in \xi_i)(\forall k \neq i)(\exists a_k \in \xi_k) (a_1, \dots, a_n) \in p^M.$$

Example 4. Consider the SD-extension \mathcal{IR} of Σ^1 -model \mathcal{R} (Σ^1 as defined in Example 1, and \mathcal{R} as defined in Example 2). Let R_0 be a finite subset of the set of all real numbers with two additional elements: $-\infty$ and $+\infty$. An R_0 -bounded interval is a set

$$\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{x \in \text{real}^{\mathcal{R}} \mid \underline{\mathbf{x}} \leq x \leq \overline{\mathbf{x}}\},$$

where $\underline{\mathbf{x}}, \overline{\mathbf{x}} \in R_0$. The set of all R_0 -bounded intervals is finite and closed under intersection:

$$\mathbf{x} \cap \mathbf{y} = [\max(\underline{\mathbf{x}}, \underline{\mathbf{y}}), \min(\overline{\mathbf{x}}, \overline{\mathbf{y}})].$$

It is easy to see that $\text{real}^{\mathcal{R}} = [-\infty, +\infty]$, and $\emptyset = [x, y]$ for any $x > y$. Let $\text{real}^{\mathcal{IR}}$ be the set of all R_0 -bounded intervals (this set satisfies all properties of a definition of an SD-extension). For any real x we define x^+ and $x^- \in R_0$ as follows:

$$\begin{aligned} x^+ &= \inf\{y \in R_0 \mid x \leq y\}, \\ x^- &= \sup\{y \in R_0 \mid y \leq x\}. \end{aligned}$$

Let $R \subseteq \text{real}^{\mathcal{R}}$, then $\text{real}^{\mathcal{IR}}[R] = [(\inf R)^-, (\sup R)^+]$. Consider the functions $+^{\mathcal{IR}}, *^{\mathcal{IR}} : \text{real}^{\mathcal{IR}} \rightarrow \text{real}^{\mathcal{IR}}$. We have (see (4))

$$\begin{aligned} \mathbf{x} +^{\mathcal{IR}} \mathbf{y} &= [(\underline{\mathbf{x}} +^{\mathcal{R}} \underline{\mathbf{y}})^-, (\overline{\mathbf{x}} +^{\mathcal{R}} \overline{\mathbf{y}})^+], \\ \mathbf{x} *^{\mathcal{IR}} \mathbf{y} &= [\min((\underline{\mathbf{x}} *^{\mathcal{R}} \underline{\mathbf{y}})^-, (\underline{\mathbf{x}} *^{\mathcal{R}} \overline{\mathbf{y}})^-, (\overline{\mathbf{x}} *^{\mathcal{R}} \underline{\mathbf{y}})^-, (\overline{\mathbf{x}} *^{\mathcal{R}} \overline{\mathbf{y}})^-), \\ &\quad \max((\underline{\mathbf{x}} *^{\mathcal{R}} \underline{\mathbf{y}})^+, (\underline{\mathbf{x}} *^{\mathcal{R}} \overline{\mathbf{y}})^+, (\overline{\mathbf{x}} *^{\mathcal{R}} \underline{\mathbf{y}})^+, (\overline{\mathbf{x}} *^{\mathcal{R}} \overline{\mathbf{y}})^+)]. \end{aligned}$$

The predicates are defined as follows (see (5)):

$$\begin{aligned} \mathbf{x} =^{\mathcal{IR}} \mathbf{y} &\text{ iff } \underline{\mathbf{x}} =^{\mathcal{R}} \underline{\mathbf{y}} \text{ and } \overline{\mathbf{x}} =^{\mathcal{R}} \overline{\mathbf{y}}, \\ \mathbf{x} \leq^{\mathcal{IR}} \mathbf{y} &\text{ iff } \underline{\mathbf{x}} \leq^{\mathcal{R}} \underline{\mathbf{y}} \text{ and } \overline{\mathbf{x}} \leq^{\mathcal{R}} \overline{\mathbf{y}}. \end{aligned}$$

The definitions of $+_i^{\mathcal{IR}}$, $*_i^{\mathcal{IR}}$, $=_i^{\mathcal{IR}}$, and $\leq_i^{\mathcal{IR}}$ ($i = 1, 2$) are left to a reader.

Estimates of variables in a subdefinite extension of a model have some remarkable properties. We discuss them in the following

Definition 10. Given a many-sorted signature $\Sigma = (S, F, P)$, a Σ -model M , its SD-extension, a Σ -model $*M$, and an S -sorted set of variables X , an estimate of variables of X in $*M$ is called a *subdefinite estimate* (an SD-estimate).

It is easy to see that the set of SD-estimates $X \rightarrow *M$ is a *partially ordered* set $(X \rightarrow *M, \subseteq)$ with $\phi \subseteq \psi$ iff $\phi_s(x) \subseteq \psi_s(x)$ for any $s \in S$, $x \in X_s$. If $\phi \subseteq \psi$ and $\psi \subseteq \phi$, then $\phi = \psi$. Obviously, *the greatest element* of the set of all estimates is $\lambda : X \rightarrow *M$, such that $\lambda_s(x) = s^M$ for all $s \in S$, $x \in X_s$.

For $v : X \rightarrow M$ and $\phi : X \rightarrow *M$ we will write $v \in \phi$ iff $v_s(x) \in \phi_s(x)$ for all $s \in S$, $x \in X_s$. Also, for $\phi, \psi : X \rightarrow *M$, we define $\phi \cap \psi : X \rightarrow *M$ as $(\phi \cap \psi)_s(x) = \phi_s(x) \cap \psi_s(x)$ for all $s \in S$, $x \in X_s$.

We will apply the notion of SD-consistency to an SD-estimate. Let C be a $\Sigma(X)$ -constraint in form $p(x_1, \dots, x_n)$ (where $p \in P_{s_1 \dots s_n}$, $x_i \in X_{s_i}$ for $i = 1, \dots, n$). An SD-estimate $\phi : X \rightarrow *M$ is *SD-consistent* w.r.t. C iff the SD-values $\phi_{s_1}(x_1), \dots, \phi_{s_n}(x_n)$ are consistent w.r.t. p^M , i. e. (see Proposition 4)

$$(\phi_{s_1}(x_1), \dots, \phi_{s_n}(x_n)) \in p^{*M}.$$

4. DENOTATIONAL SEMANTICS OF CONSTRAINT PROPAGATION

In the sequel, we suppose a many-sorted signature $\Sigma = (S, F, P)$, a $\Sigma(X)$ -constraint satisfaction problem (2), a Σ -model M and its SD-extension, a Σ -model $*M$ to be given. Let us denote by \mathcal{C} the set of all atoms (constraints) of (2).

A fundamental notion of constraint propagation in subdefinite models is a notion of *interpretation*, or *filtering* of a constraint. The interpretation of a constraint is a mapping on the set of SD-estimates $X \rightarrow *M$. The purpose of such a mapping is the filtering of those values of SD-estimate of variables which are SD-consistent with the constraint. We regard an SD-estimate of variables as a set of possible witnesses of the realization of constraint satisfaction problem (2) in M . If SD-estimate contains some estimates which are not witnesses of CSP, we try to exclude them from the

SD-estimate by filtering of each constraint of CSP. The process of filterings (which is called *constraint propagation* and will be discussed below) is very similar to that of achieving *local consistency* proposed in [1]. The result of the process is an SD-estimate, which is SD-consistent with each constraint of CSP. In this section we define the result of the process in terms of the greatest fixed point of such filterings, i. e. we define denotational semantics of constraint propagation.

Definition 11. Given a $\Sigma(X)$ -constraint C , we define a mapping

$$I_C : (X \rightarrow {}^*M) \rightarrow (X \rightarrow {}^*M)$$

(which will be called an *interpretation*, or *filtering* of C) as follows. For an SD-estimate $\phi : X \rightarrow {}^*M$

$$I_C(\phi) = \psi,$$

where

1. $\psi \subseteq \phi$,
2. ψ is SD-consistent w.r.t. C ,
3. ψ is the maximal SD-estimate from those satisfying the properties 1 and 2.

Proposition 5. *The above definition is correct, i. e. $I_C(\phi)$ is defined for any $\phi : X \rightarrow {}^*M$.*

Proof. Let $C \equiv p(x_1, \dots, x_n)$ for $p \in P_w$, $w = s_1 \dots s_n \in S^+$. Consider ψ defined as follows:

$$\begin{aligned} \psi_s(x) &= \phi_s(x) \text{ for all } x \in X_s \setminus \{x_1, \dots, x_n\}, s \in S, \\ \psi_{s_i}(x_i) &= s_i^{*M}[\pi_i(\phi_{s_1}(x_1) \times \dots \times \phi_{s_n}(x_n) \cap p^M)], \quad i = 1, \dots, n. \end{aligned}$$

To simplify notations, define

$$\begin{aligned} \alpha &= (\phi_{s_1}(x_1), \dots, \phi_{s_n}(x_n)), \\ \beta &= (\psi_{s_1}(x_1), \dots, \psi_{s_n}(x_n)). \end{aligned}$$

We have $\beta = w^{*M}[\alpha \cap p^M]$ (see Proposition 3). Since $\alpha \cap p^M \subseteq \alpha$ and $w^{*M}[\dots]$ is monotone w.r.t. set inclusion, $\beta = w^{*M}[\alpha \cap p^M] \subseteq w^{*M}[\alpha] = \alpha$. Therefore, $\psi \subseteq \phi$.

We have $\alpha \cap p^M \subseteq w^{*M}[\alpha \cap p^M] = \beta \subseteq \alpha$. Therefore, $\alpha \cap p^M = \beta \cap p^M$, and $w^{*M}[\beta \cap p^M] = w^{*M}[\alpha \cap p^M] = \beta$, i. e. β is SD-consistent w.r.t. p^M (see Proposition 4). We have just proved that ψ is SD-consistent w.r.t. C .

Let $\theta \subseteq \phi$ be an SD-consistent estimate. Define $\gamma_i = \theta_{s_i}(x_i)$ (for $i = 1, \dots, n$), and $\gamma = (\gamma_1, \dots, \gamma_n)$. We have $\gamma \subseteq \alpha$, therefore

$$\gamma = w^{*M}[\gamma \cap p^M] \subseteq w^{*M}[\alpha \cap p^M] = \beta.$$

Since $\theta_s(x) \subseteq \phi_s(x)$ holds for all $s \in S$ and $x \in X_s \setminus \{x_1, \dots, x_n\}$, we can finally say that $\theta \subseteq \psi$.

The following proposition states the remarkable properties of a filtering.

Proposition 6. *Each filtering $I_C : (X \rightarrow *M) \rightarrow (X \rightarrow *M)$ for $C \in \mathcal{C}$ satisfies the following conditions for all $\phi, \psi : X \rightarrow *M$:*

1. *correctness: if $v : X \rightarrow M$ is a witness of (2) in M and $v_s(x) \in \phi_s(x)$ for all $s \in S$, $x \in X_s$ then $v_s(x) \in I_C(\phi)_{s_1}(x)$,*
2. *contractance: $I_C(\phi) \subseteq \phi$,*
3. *monotonicity: if $\phi \subseteq \psi$ then $I_C(\phi) \subseteq I_C(\psi)$,*
4. *idempotency: $I_C(I_C(\phi)) = I_C(\phi)$.*

Proof. Let $C \equiv p(x_1, \dots, x_n)$ for $p \in P_w$, $w = s_1 \dots s_n \in S^+$. Let $a = (v_{s_1}(x_1), \dots, v_{s_n}(x_n))$, $\alpha = (\phi_{s_1}(x_1), \dots, \phi_{s_n}(x_n))$. We have $a \in p^M$ and $a \in \alpha$, consequently,

$$a \in \alpha \cap p^M \subseteq w^{*M}[\alpha \cap p^M] = (I_C(\phi)_{s_1}(x_1), \dots, I_C(\phi)_{s_n}(x_n)).$$

Therefore, $v \in I_C(\phi)$.

The mapping $I_C(\dots)$ is monotone since the mappings $\dots \cap p^M$ and $w^{*M}[\dots]$ are monotone.

The properties of contractance and idempotency are simple consequences of Definition 4.

The main result of this section is the following

Proposition 7. *The following assertions are true:*

1. *There exists the maximal estimate ϕ^* , which is SD-consistent w.r.t. each $C \in \mathcal{C}$.*
2. *If $v : X \rightarrow M$ is a witness of the realization of CSP (2) in M , then $v \in \phi^*$.*

Proof.

1. It is easy to see that an SD-estimate ϕ is SD-consistent w.r.t. a constraint C iff $I_C(\phi) = \phi$, i. e. ϕ is a fixed point of the mapping I_C . Therefore, ϕ^* (if it does exist) should be the maximal (the greatest) common fixed point of the system of mappings $\mathcal{I} = \{I_C \mid C \in \mathcal{C}\}$. We will prove the existence of this fixed point. Let us consider a mapping $I = I_{C_1} \circ \dots \circ I_{C_n}$ (where $\{C_1, \dots, C_n\} = \mathcal{C}$). Since each filtering is a contractive mapping, we have $\lambda \supseteq I(\lambda) \supseteq I^2(\lambda) \supseteq \dots$

(recall that λ is the greatest element of the set of all estimates). Since all the carriers of Σ -model *M are finite sets, then the set of all estimates of variables from X in *M is finite, too. Therefore, there exists $N > 0$ such that $I^N(\lambda) = I^{N+1}(\lambda)$. Let $\phi^* = I^N(\lambda)$. Obviously, ϕ^* is a common fixed point of \mathcal{I} . Let ϕ be a common fixed point of \mathcal{I} . We have $\phi \subseteq \lambda$ and $\phi = I^N(\phi) \subseteq I^N(\lambda) = \phi^*$ (since each filtering is a monotone mapping, see Proposition 6), i. e. ϕ^* is the greatest common fixed point of \mathcal{I} .

2. For each witness $v : X \rightarrow M$ of the realization of CSP (2) in M we have $v \in \lambda$. Since all filterings are correct (see Proposition 6), we have $v \in \phi^*$.

The SD-estimate ϕ^* defined in the last proposition is a result of constraint propagation in subdefinite models. It defines denotational semantics of the process. Proposition 7 suggests a method for computation of ϕ^* (which is a kind of the method of successive approximations), but it is not effective. Now we consider a better computation algorithm.

5. OPERATIONAL SEMANTICS

In this section we consider operational semantics of constraint propagation in subdefinite models. The process will be described in terms of *states* and *transition rules* of *abstract machine*, then we will prove its termination and correctness (equivalence of both denotational and operational semantics). Informally, a state is an SD-estimate of variables with a set of *active* constraints. We regard an SD-estimate as information about single (precise) values of variables of CSP, i. e. information about all witnesses of the realization of CSP in a model M . Going from state to state, we try to get more information about the witnesses of CSP (to make an SD-estimate more definite). The transitions between states are performed by filterings of active constraints. Initially, all constraints of CSP are active. After filtering of constraint, it is passive, but all constraints containing variables which have changed their SD-values during the filtering will be placed in the set of active ones. The process is terminated when there are no active constraints. Let us consider these notions formally.

Definition 12. Let $\Sigma = (S, F, P)$ be a many-sorted signature, X be an S -sorted set of variables, \mathcal{C} be a set of $\Sigma(X)$ -constraints defining some constraint satisfaction problem, and *M be an SD-extension of a Σ -model M . We define an engine for computing the greatest estimate of variables X in *M , which is SD-consistent w.r.t. \mathcal{C} , as an *abstract machine* with *states*

and *transition rules*.

A *state* is a pair (ϕ, Q) , where ϕ is an estimate of variables X in *M and $Q \subseteq \mathcal{C}$. We define *transitions* from a state to a state by means of the binary relation \vdash over the set of all states $S = (X \rightarrow {}^*M) \times \mathcal{P}(\mathcal{C})$ as follows: $(\phi, Q) \vdash (\psi, R)$ iff

- $Q \neq \emptyset$,
- $\psi = I_C(\phi)$ for some $C \in Q$,
- $R = Q \setminus \{C\} \cup \{C' \in \mathcal{C} \mid (\exists x \in \text{vars}_s(C')) \phi_s(x) \neq \psi_s(x)\}$, where $\text{vars}(C)$ is an S -sorted set of variables of a constraint C .

A state (λ, \mathcal{C}) , where $\lambda_s(x) = s^M$ for all $s \in S$, $x \in X_s$ will be called an *initial state*, any state (ϕ, \emptyset) will be called a *final state*. We will also write $(\phi, Q) \vdash^n (\psi, R)$, iff there exists a sequence of states $(\phi_0, Q_0), \dots, (\phi_n, Q_n)$ such that $(\phi, Q) = (\phi_0, Q_0) \vdash \dots \vdash (\phi_n, Q_n) = (\psi, R)$. We will write $(\phi, Q) \vdash^* (\psi, R)$, iff there exists $n \geq 0$ such that $(\phi, Q) \vdash^n (\psi, R)$.

At the beginning, the machine is at the state (λ, \mathcal{C}) . At each state (ϕ, Q) , the machine performs a nondeterministic choice of $C \in Q$ and makes a transition to the state $(I_C(\phi), R)$ (R is defined above). If $Q = \emptyset$, the machine stops.

The following assertion states the link between operational and denotational semantics of constraint propagation in subdefinite models.

Proposition 8. *In the terms defined above, the following assertions are true:*

1. *There exists $N > 0$ such that if $(\lambda, \mathcal{C}) \vdash^N (\phi, R)$, then (ϕ, R) is the final state, i. e. $R = \emptyset$.*
2. *If $(\lambda, \mathcal{C}) \vdash^* (\phi, \emptyset)$, then ϕ is the greatest estimate of variables X in *M , which is SD-consistent w.r.t. \mathcal{C} , i. e. $\phi = \phi^*$.*

Proof. If $(\phi, Q) \vdash (\psi, R)$, then $\psi \subseteq \phi$ and if $\psi = \phi$, then $\#R = \#Q - 1$ (since no variable has changed a value). Consider a sequence

$$(\lambda, \mathcal{C}) = (\phi_0, Q_0) \vdash (\phi_1, Q_1) \vdash \dots \vdash (\phi_n, Q_n) \vdash \dots$$

We have $\phi_0 \supseteq \phi_1 \supseteq \dots \supseteq \phi_n \supseteq \dots$. If $\phi_i = \phi_j$ for $j > i$, then $\#Q_j = \#Q_i - (j - i)$. If $Q_j \neq \emptyset$, then $j - i < \#Q_i \leq \#\mathcal{C}$. And the longest sequence $\lambda = \phi_0 > \phi_{i_1} > \phi_{i_2} > \dots > \phi_{i_L} = \mu$ (where $\mu_s(x) = \emptyset$ for all $s \in S$, $x \in X_s$) has the length $L = \sum_{s \in S} \sum_{x \in X_s} l(s^{*M})$, where $l(s^{*M})$ is the length of the maximal decreasing sequence of elements from s^{*M} : $s^M = \alpha_0 \supset \alpha_1 \supset \dots \supset \alpha_{l(s^{*M})} = \text{set}$. Is is obvious that $l(s^{*M}) \leq \#s^{*M}$.[‡]

[‡]The estimation $\#s^{*M}$ is very rough for $l(s^{*M})$, and we can define it more precisely for SD-extensions discussed in the following section.

Therefore there exists $N > 0$ such that if $(\lambda, \mathcal{C}) \vdash^N (\phi, R)$, then $R = \emptyset$. Moreover, we can estimate N as follows: $N \leq \#\mathcal{C} \times \sum_{s \in S} \sum_{x \in X_s} l(s^{*M}) \leq \#\mathcal{C} \times \sum_{s \in S} \sum_{x \in X_s} \#s^{*M}$.

To prove the second assertion of the proposition, we note that if $(\lambda, \mathcal{C}) \vdash^* (\phi, Q)$, then $I_C(\phi) = \phi$ for all $C \in \mathcal{C} \setminus Q$ (it can be easily proved by induction over the length of the chain of transitions). Therefore, if $(\lambda, \mathcal{C}) \vdash^* (\phi, \emptyset)$, then ϕ is a fixed point of all I_C ($C \in \mathcal{C}$). Since each of I_C is a monotone mapping, then ϕ is the greatest fixed point (proposition 7), i. e. $\phi = \phi^*$.

In the programming technology based on SD-models, one uses some strategies to perform a choice at each state. One of the strategies is to assign a (static or dynamic) priority to each constraint $C \in \mathcal{C}$ and to choose a constraint with maximal priority from Q at state (ϕ, Q) . Asynchronous parallel processing of such transitions is normally used, but these aspects of SD-models are out of the focus of this paper.

And now we can define the notion of a *subdefinite model* which we have used above informally.

Definition 13. A *subdefinite model* (or briefly, *SD-model*) is a tuple

$$P = (\Sigma, X, \Phi, M, *M, A),$$

where

- $\Sigma = (S, F, P)$ is a many-sorted signature,
- X is an S -sorted set of variables,
- \mathcal{C} is a $\Sigma(X)$ -constraint satisfaction problem,
- M is a Σ -model,
- $*M$ is its SD-extension, and
- A is an abstract machine which computes the greatest SD-consistent w.r.t. \mathcal{C} estimate of variables X in $*M$, containing all witnesses of the realization of \mathcal{C} in M .

We conclude this section with the following simple example of a subdefinite model.

Example 5. Consider the signature Σ^1 defined in Example 1, and the following Σ^1 -existential conjunction:

$$(\exists x, y \in \text{real}) \ x + y = 6, \ 2 * x = y, \ -100 \leq x, \ x \leq 100, \ -100 \leq y, \ y \leq 100,$$

which can be transformed (see Proposition 1) into the following CSP:

$$(\exists x, y, t_1, t_2, t_3, t_4, t_5, t_6 \in \text{real}) \ x + y = t_1, \ t_2 * x = y, \ 6 = t_1, \ 2 = t_2$$

$$\begin{aligned}
-100 &= t_3, \quad 100 = t_4, \quad t_3 \leq x, \quad x \leq t_4, \\
t_5 &\leq y, \quad y \leq t_6, \quad -100 = t_5, \quad 100 = t_6
\end{aligned}$$

Let us try to find all witnesses of the realization of this CSP in the Σ^1 -model \mathcal{R} defined in Example 2. Let R_0 be the finite set of rational numbers with $-\infty$ and $+\infty$:

$$R_0 = \{-\infty, -100, -99.9, -99.8 \dots, 99.9, 100, +\infty\}.$$

Consider SD-extension \mathcal{IR} of \mathcal{R} by R_0 -bounded intervals discussed in Example 4. The filterings of constraints $I_{x+y=t_1}$, $I_{t_2*x=y}$, $I_{6=t_1}$, $I_{2=t_2}$, $I_{t_3 \leq x}$, $I_{x \leq t_4}$, $I_{t_5 \leq y}$, $I_{y \leq t_6}$, $I_{-100=t_3}$, $I_{100=t_4}$, $I_{-100=t_5}$ and $I_{100=t_6}$ are defined via operations over subdefinite universes which have been discussed in Example 4. The result of constraint propagation (which terminates in 79 steps) is the following:

$$\begin{aligned}
x &= [1.9, 2.1], \\
y &= [3.9, 4.1], \\
t_1 &= [6, 6], \\
t_2 &= [2, 2], \\
t_3 &= [-100, -100], \\
t_4 &= [100, 100], \\
t_5 &= [-100, -100], \\
t_6 &= [100, 100].
\end{aligned}$$

This SD-estimate contains a single witness of the realization of this CSP in M :

$$x = 2, \quad y = 4.$$

6. TYPES OF SD-EXTENSIONS

It should be noted that every model can have several possible SD-extensions that differ in their inference/computation power as well as in the computer resources they require. Let us consider some versions of such the extensions which have been successfully used in the mathematical problem solver *UniCalc* [12], the intelligent solver of arithmetic puzzles *ARBOOZ* [8],

the system for calendar scheduling *Time-Ex* [13], and the technological programming environment *NeMo-TeC* [5]. To simplify the notation, let us denote the set of values M_s of the sort s in the model M by U , and call this set a *universe*. We suppose that U_0 is a finite subset of U . The class of all possible SD-extensions of the universe U will be denoted by $SD(U)$.

1) The simplest SD-extension of the universe U is the SD-extension *Single* which has the following form:

$$U^{Single} = \{\emptyset\} \cup \{U\} \cup \{\{x\} | x \in U_0\}.$$

It is clear that U^{Single} conforms to the definition of an SD-extension; it is obtained by adding two special elements, *undefined* (U) and *contradiction* (\emptyset), to the set U_0 .

2) The maximal SD-extension of U , which we denote by U^{Enum} , is the set containing U and all subsets of U_0 , that is:

$$U^{Enum} = \mathcal{P}(U_0) \cup \{U\}.$$

In the case when U is a lattice (a set with two associative and commutative operations \vee and \wedge satisfying the absorption law and the idempotent law), it is possible to define such types of SD-extensions of U as *intervals* and *multi-intervals*. Let U_0 be a finite sublattice of U , $-\infty$ and $+\infty$ be minimal and maximal elements of U , respectively (if they do not exist in U , we add them to U with $\forall x \in U \cup \{-\infty, +\infty\} \quad x \wedge -\infty = -\infty, \quad x \vee +\infty = +\infty$).

3) the SD-extension by intervals:

$$U^{Interval} = \{[\underline{x}, \bar{x}] \mid \underline{x}, \bar{x} \in U_0 \cup \{-\infty, +\infty\}\}.$$

Here $[\underline{x}, \bar{x}] = \{x \in U \mid \underline{x} \wedge x = \underline{x} \text{ and } x \vee \bar{x} = \bar{x}\}$, \underline{x} is called a *lower bound* of the interval, and \bar{x} is an *upper* one. It is obvious that

$$[\underline{x}, \bar{x}] \cap [\underline{y}, \bar{y}] = [\underline{x} \vee \underline{y}, \bar{x} \wedge \bar{y}].$$

Here, the empty set is represented by an interval $[\underline{x}, \bar{x}]$, where $\underline{x} \vee \bar{x} \neq \bar{x}$. The entire universe U is represented by the interval $[-\infty, +\infty]$, and the single element by $\{x\} = [x, x]$.

4) the SD-extension by multi-intervals:

$$U^{Multi} = \{\xi \mid \xi = \cup \xi_i, \xi_i \in U^{Interval}, i = 1, 2, \dots\}.$$

Each multi-interval is a finite set of intervals. It is obvious that \emptyset , U , and $\{x\}$ are represented exactly as in the case of intervals, and

$$\alpha \cap \beta = \{\alpha_i \cap \beta_i | i = 1, 2, \dots, j = 1, 2, \dots\}.$$

We have discussed the interval extension of the set of all real numbers in Example 4. (The notion of the interval extension is well-known, see [14].) Let us consider another application of intervals.

Example 6. Let $V = \mathcal{P}(U)$ for some universe U , i. e. elements of V are subsets of U . It is obvious that V is a lattice with $x \wedge y = x \cap y$, $x \vee y = x \cup y$. Let $V_0 = \mathcal{P}(U_0)$, where U_0 is a finite subset of U , $-\infty = \emptyset$, $+\infty = U$. Then $V^{Interval}$ is an SD-extension of the universe $\mathcal{P}(U)$.

These sets have been firstly proposed in [4], where they are called *subdefinite sets*. Consider semantics of an interval $[\underline{x}, \bar{x}]$, where $\underline{x}, \bar{x} \subseteq U$. The elements of \underline{x} *must* belong to the set represented by the interval, the elements of \bar{x} *may* belong to the set.

Consider the following function computing the cardinal number of a set: $\# : V \rightarrow \mathbf{N}$. Its SD-extension $\#\# : V^{Interval} \rightarrow \mathbf{N}^{Interval}$ is defined as follows: $\#\#[\underline{x}, \bar{x}] = [\#\underline{x}, \#\bar{x}]$.

The common properties of intervals in Boolean algebra have been considered in [15], where another examples of interval set operations can be also found.

5) *Structural SD-extension.* Consider now a universe defined as the Cartesian product of several sets: $U = U_1 \times \dots \times U_n$. We can apply the SD-extensions U^{Single} and U^{Enum} to U . Moreover, if each of U_i is a lattice, then U is a lattice, too (as the Cartesian product of lattices), and so we can apply the SD-extensions $U^{Interval}$ and U^{Multi} to U as well. However, one extra SD-extension of U can be proposed.

Let $*U_i \in SD(U_i), i = 1, \dots, n$. Then the system $*U_1 \times \dots \times *U_n$ will satisfy the properties of the subdefinite extension, too, that is, $*U_1 \times \dots \times *U_n \in SD(U_1 \times \dots \times U_n)$. The following question is of interest: if the form of the SD-extension, applicable to both U_i and U , is fixed, then what will be the relationship between the set systems $*(U_1 \times \dots \times U_n)$ and $*U_1 \times \dots \times *U_n$? Consider these extensions for each type of the SD-extensions discussed above:

$$\begin{aligned} (U_1 \times \dots \times U_n)^{Single} &\subseteq U_1^{Single} \times \dots \times U_n^{Single}, \\ (U_1 \times \dots \times U_n)^{Enum} &\supseteq U_1^{Enum} \times \dots \times U_n^{Enum}, \\ (U_1 \times \dots \times U_n)^{Interval} &= U_1^{Interval} \times \dots \times U_n^{Interval}, \end{aligned}$$

$$(U_1 \times \dots \times U_n)^{Multi} \supseteq U_1^{Multi} \times \dots \times U_n^{Multi}.$$

Thus we see that the choice of the SD-extension in the form $*(U_1 \times \dots \times U_n)$ or $*U_1 \times \dots \times *U_n$ is not important only for intervals. Note that in the case of enumerations (*Enum*) and multi-intervals, the choice of a “richer” representation (in the form $*(U_1 \times \dots \times U_n)$) considerably increases the resources necessary to store these subdefinite expressions.

7. CONCLUSION

A framework for solving constraint satisfaction problem, namely, a subdefinite model apparatus, is presented in this paper. This approach takes into account some partially known information about values of objects.

Like other constraint propagation techniques, SD-models are based on local computations; however, computations on all SD-models are performed by a single data-flow algorithm which does not depend on the type of the original problem. Generality of the approach, which is the basis of SD-models, makes it possible to use them for the solution of problems which are traditionally placed in different classes. For example, the apparatus of SD-models can be applied to numerical problems (systems of linear and nonlinear equations, or inequalities over integer and real variables), to logical and combinatorial problems, to problems on sets, etc. The most remarkable fact is that all these problems can be solved simultaneously within a single SD-model.

Several programming systems based on the technology of the SD-models have been implemented. The *NeMo+* object-oriented technological environment [16] is the most general of them.

The investigation of the subdefiniteness and the use of the SD-models in constraint programming is in progress. In particular, our objectives for the nearest future are further development of the methods of representating the SD-extensions with the corresponding modification of the computation process and extension of the notion of subdefinite models to object-oriented subdefinite models.

ACKNOWLEDGEMENTS

The author is indebted to professor Alexandre Zamulin for fruitful discussions and helpful comments on previous versions of the paper. Natalia Cheremnykh has pleasantly read the draft version of the paper and cor-

rected a few mistakes in English sentences. Finally, I would like to thanks Vitaly Telerman for his support and encouragement.

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**НЕКОТОРЫЕ ФОРМАЛЬНЫЕ АСПЕКТЫ
НЕДООПРЕДЕЛЕННЫХ МОДЕЛЕЙ**

**Препринт
49**

Рукопись поступила в редакцию 09.02.98

Рецензент А. В. Замулин

Редактор Н. А. Черемных

Подписано в печать 20.03.98

Формат бумаги 60×84 1/16

Тираж 100 экз.

Объем 1,4 уч.-изд.л., 1,5 п.л.

Отпечатано на ризографе "AL Group", 630090, пр. Акад. Лаврентьева, 6