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THE POLYMODAL LOGIC BASED ON A-SPACES

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The polymodal logic based on A-spaces is considered. Modalities connected with A-spaces are interpreted as tense ones. The nearest and detached future or past are considered, i.e. four modalities are used. The tense modalities are especially important for studying the parallel computations and for a verification of programs. A set of axioms is described and a theorem of correctness is formulated for these axioms relative to frames based on A - spaces. Logic calculus L' is introduced, for that some theorem about a canonical model is formulated. A connection between fragments of L' and the well known logic S_4 is established. The proofs are omitted in view of their cumbersomity.

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ПОЛИМОДАЛЬНАЯ ЛОГИКА НА ОСНОВЕ A-ПРОСТРАНСТВ

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Рассматривается полимодальная логика на основе A-пространств. Модальности, связанные с A-пространствами, интерпретируются как временные. Речь идет о ближайшем и отдаленном будущем или прошлом, т.е. используются четыре модальности. Временные модальности особенно важны при изучении параллельных вычислений и верификации программ. Описано множество аксиом и сформулирована теорема корректности для этих аксиом относительно шкал на основе A-пространств. Введено некоторое логическое исчисление L', для которого сформулирована теорема о канонической модели. Устанавливается связь между фрагментами L' и известной логикй S_4. Доказательства опущены ввиду их громоздкости.

INTRODUCTION

The polymodal logic based on A-spaces is considered. The conception of A-space was introduced by Ershov Yu.L. [1] with the purpose to develop the theory of computable functionals [2] and investigate continuous lattices considered by Scott D. [3] in connection with the analysis of data types.

Various modalities having a form of necessity or capability are studied in modal logics [4]. In other words, a formalization of some reasonings present in the natural language and thinking of the people is carried out. Doubtlessly, it is very interesting. Note in addition, that in researches on nonclassical logics, including the modal ones, topological methods [S5] are widely used.

Recently, investigations on modal logics are developing, the source of which are problems arising in computer science. Here, it is possible to mention various works connected with methods of representation of the knowledges [6], logics of arrows [7], a refinement of conception of choice [8] and tense logics [9].

The last theme is especially important in connection with development of parallel computations [10]. Various subsystems of computing system can use different timers (it is possible to tell that each subsystem exists in the own time). There are appeared the various problems of synchronization. A relativity of time and the tight connection of time with the information is revealed in the full measure.

Note that modalities connected with A-spaces investigated in the paper are interpreted as tense ones. The tense operators of nearest and remote future or past, i.e. four modalities are used.

In the paper, a set of axioms is described and a correctness theorem is formulated for these axioms relative to frames based on A-spaces. Logic calculus L' is introduced, for that some theorem about a canonical model is formulated. A connection between fragments of L' and the well known logic S_4 is established. The proofs are omitted in view of their cumbersomity.

1. DEFINITIONS AND NOTATIONS

Let $\langle X, \tau \rangle$ be a topological t_0-space where τ is a class of all open sets.

A topological space X is called a t_0-space, if there exists an open set for each pair of different elements in X containing only one of them [5].

We shall introduce a partial order $\leq$ on X defined as follows [1]

$$\forall x, y \in X \ (x \leq y \longleftrightarrow \forall U \in \tau \ (x \in U \longrightarrow y \in U)).$$

For elements $x, y \in X$ let $x \cup y$ denote the least upper bound. We do not require that $x \cup y$ allways exist.

For any $x \in X$ let x^* denote the interior of the set $\breve{x}$ (i.e., $x^* = Int\breve{x}$) where $\breve{x} = \{y : \ y \in X, x \leq y\}$.

We say y is strongly less than x written $y \prec x$ if $x \in y^*$.

Notice some simple properties [1] of the relations $\leq$ and $\prec$.

Remark 1.1

a) $x \leq y \ \& \ y \prec z \Longrightarrow x \prec z$;

b) $x \prec y \ \& \ y \leq z \Longrightarrow x \prec z$;

c) $x \prec y \Longrightarrow x \leq y$;

d) $x \prec z \ \& \ y \prec z \ \& \ (t = x \cup y) \Longrightarrow t \prec z$;

e) $x_0 \prec y_0 \ \& \ x_1 \prec y_1 \ \& \ (t_x = x_0 \cup x_1) \ \& \ (t_y = y_0 \cup y_1) \Longrightarrow t_x \prec t_y$.

Let $X_0 \subseteq X$, then a pair $\langle X_0, \leq \rangle$ (i.e. a restriction of the order $\leq$ to X_0) is said to be subparus of $\langle X, \leq \rangle$ if for any $x, y \in X_0$ the existence of their upper bound z in X (i. e., $x \leq z \ \& \ y \leq z$) implies the existence of their least upper bound in X_0.

Definition. A topological t_0-space X is called a A-space, if there is $X_0 \subseteq X$ satisfying the following conditions:

1) $\langle X_0, \leq \rangle$ is subparus;

2) the class of sets $\{\emptyset\} \cup \{x^* : \ x \in X_0\}$ is the basis of a topology of X ;

3) $\forall x_0 \in X_0 \ \forall x \in X \ (x_0 \prec x \Longrightarrow$
$\exists x_1 \in X_0 \ (x_0 \prec x_1 \ \& \ x_1 \prec x \ ) \ )$.

It has been noted in the paper [1] that if the condition 1) is realized, then conditions 2) and 3) are equivalent to the following proposition:

4) $\forall U \in \tau \forall x \in U \exists x_0 \in X_0 \cap U (x_0 \prec x)$.

2. MODALITIES INDUCED BY ORDERS

Let X be a t_0-space with the relation $\leq$ and $\prec$ introduced above. A 3-tuple $\langle X, \ \leq, \ \prec \rangle$ will be called the frame of t_0-space. If X is A-space, then $\langle X, \ \leq, \ \prec \rangle$ will be called the frame of A-space.

Our goal is the description and the investigation of the modal logic using the frame of the given form for building the semantics analogous to the Kripke models. In this work, the propositional calculus is considered.

The language of a modal propositional logic contains :

a) propositional variables $p_1,\ p_2,\ldots$;
b) logical connectives $\&,\ \vee,\ \longrightarrow,\ \neg$;
c) modal operators $\Diamond_{\leq},\ \Diamond_{\prec},\ \Box_{\leq},\ \Box_{\prec},\ \Diamond^*_{\leq},\ \Diamond^*_{\prec},\ \Box^*_{\leq},\ \Box^*_{\prec}$;
d) constant β.

The formulas of the language are then defined inductively as usually [4]. One additional rule for constructing the formulas is added:

$$\text{if } A \text{ is a formula, then } \Diamond_{\leq}A,\ \Diamond_{\prec}A,\ \Box_{\leq}A,\ \Box_{\prec}A\ ,$$

$$\Diamond^*_{\leq}A,\ \Diamond^*_{\prec}A,\ \Box^*_{\leq}A,\ \Box^*_{\prec}A \text{ are formulas.}$$

Modalities $\Diamond_{\leq},\ \Diamond_{\prec},\ \Box_{\leq},\ \Box_{\prec}$ interpret as a necessity and a possibility in a future.

Modalities $\Diamond^*_{\leq},\ \Diamond^*_{\prec},\ \Box^*_{\leq},\ \Box^*_{\prec}$ interpret as a necessity and a possibility in a detached future.

Thus the nearest and detached future or past are considered, i.e. four modalities are used.

3. ON THE SEMANTICS

For modal logics, we will consider models analogous to the Kripke models. We will say about the truth in the given moment of time, where moments of time are points of some ordered set.

Let us consider a model

$$M = \langle X,\ X_0,\ \leq,\ \prec,\ \models\rangle$$

where $\langle X,\ \leq,\ \prec\rangle$ is a frame of t_0-space, $X_0 \subseteq X$, $\models$ is a binary relation of truth between X and a set of formulas.

Such model M will be called the t_0-model if X is a t_0-space. If X is a A-space then M will be called the A-model.

The truth value of a formula A at a moment x (a notation $x \models A$ will be used) is defined by induction according to the following clauses:

We suppose $x \models p_i$ or not $x \models p_i$ for any propositional variable p_i and for any $x \in X$,

0) $x \models \beta \iff x \in X_0$,
1) $x \models \neg A \iff$ not $x \models A$,
2) $x \models A\&B \iff x \models A$ and $x \models B$,
3) $x \models A \vee B \iff x \models A$ or $x \models B$,

4) $x \models A \longrightarrow B \iff x \models A \Longrightarrow x \models B$,
5) $x \models \Diamond_{\leq} A \iff \exists y\ (x \leq y$ and $y \models A)$,
6) $x \models \Diamond_{\prec} A \iff \exists y\ (x \prec y$ and $y \models A)$,
7) $x \models \Box_{\leq} A \iff \forall y\ (x \leq y \Longrightarrow y \models A)$,
8) $x \models \Box_{\prec} A \iff \forall y\ (x \prec y \Longrightarrow y \models A)$,
9) $x \models \Diamond^{*}_{\leq} A \iff \exists y\ (y \leq x$ and $y \models A)$,
10) $x \models \Diamond^{*}_{\prec} A \iff \exists y\ (y \prec x$ and $y \models A)$,
11) $x \models \Box^{*}_{\leq} A \iff \forall y\ (y \leq x \Longrightarrow y \models A)$,
12) $x \models \Box^{*}_{\prec} A \iff \forall y\ (y \prec x \Longrightarrow y \models A)$.

A formula A is said to be hold in the model M, if for all $x \in X$ there is $x \models A$. A formula A is said to be valid in the frame, if it is hold in any model on the underlying this frame . A formula A is said to be valid, if it is valid in all frames.

Remark. The formulas $A \longrightarrow \Box^{*}_{i} \Diamond_{i} A$, $A \longrightarrow \Box_{i} \Diamond^{*}_{i} A$,
$\Box_{i}(A \longrightarrow B) \longrightarrow (\Box_{i} A \longrightarrow \Box_{i} B)$, $\Box^{*}_{i}(A \longrightarrow B) \longrightarrow (\Box^{*}_{i} A \longrightarrow \Box^{*}_{i} B)$, where $i \in \{\leq,\ \prec\}$, are valid [9].

Proposition 3.1. Let $M = \langle X,\ X_0,\ \leq,\ \prec,\ \models\rangle$ be a t_0-model, $x \in X$. Then:

1) $x \models \Diamond_{\leq} \Diamond_{\prec} A \longrightarrow \Diamond_{\prec} A$,
2) $x \models \Diamond_{\prec} \Diamond_{\leq} A \longrightarrow \Diamond_{\prec} A$,
3) $x \models \Diamond_{\prec} A \longrightarrow \Diamond_{\leq} A$,
$x \models \Box_{\leq} A \longrightarrow \Box_{\prec} A$.

Proposition 3.2. Let M be a t_0 - model. If for all $x, y \in X$ there exists $x \cup y$ then
$x \models \Diamond_{\prec} \Box_{\prec} A \& \Diamond_{\prec} \Box_{\prec} B \longrightarrow \Diamond_{\prec} \Box_{\prec} (A \& B)$.

Proposition 3.3. Let $M = \langle X,\ X_0,\ \leq,\ \prec,\ \models\rangle$ be a t_0 - model, $x \in X$. Then

1) $x \models (\Box_{\leq} A \longrightarrow A)$,
$x \models (A \longrightarrow \Diamond_{\leq} A)$,
2) $x \models (\Box_{\leq} A \longrightarrow \Box_{\leq} \Box_{\leq} A)$,
$x \models (\Diamond_{\leq} \Diamond_{\leq} A \longrightarrow \Diamond_{\leq} A)$,
3) $x \models (\Box_{\prec} A \longrightarrow \Box_{\prec} \Box_{\prec} A)$,
$x \models (\Diamond_{\prec} \Diamond_{\prec} A \longrightarrow \Diamond_{\prec} A)$.

The proof is given in [4].

4. TOPOLOGICAL PROPERTIES OF FRAMES AND CORRESPONDING AXIOMS

In this sectiom we assume that $\langle X,\ \leq,\ \prec\rangle$ is a frame of t_0 - space, $X_0 \subseteq X$. For any set $M \subseteq X$ denote $\Box_\prec M = \{x:\ \forall y(x \prec y \Longrightarrow y \in M)\}$.

Proposition 4.1.

1) If M is an open set then $M \subseteq \Box_\prec M$.

2) Proposition $M \supseteq \Box_\prec M$ is not valid in general case, even if M is an open set.

Proposition 4.2. $M \subseteq \Box_\prec M \Longleftrightarrow \forall x \in M(x^* \subseteq M)$.

Remark.

If furthermore $\forall x \in M(x \prec x)$, or in other words $\forall x \in M(x \in x^*)$, then M is open because $x \in x^* \subseteq M$.

Proposition 4.3. If $\forall U \in \tau \forall z \in U \exists x_0 \in X_0 \cap U(x_0 \prec z)$ holds in X, i. e. the condition 4) of the definition A - space is realized, then for any x:

$$x \models \Box_\prec A \longrightarrow \Box_\prec \Diamond^*_\prec(\beta \& \Box_\leq A).$$

Lemma 4.4. If X is a t_0-space, $X_0 \subseteq X$ satisfies conditions 2) and 3) of definition A-space, then X_0 is dense in X (in topological sense), i. e. for any open in X set U there exists $z_0 \in X_0 \cap U$.

Proposition 4.5. If X_0 is dense in X then for any $x \in X$

$$x \models \Box_\prec A \longrightarrow \Diamond_\leq(\beta\ \&\ \Box_\leq A).$$

Proposition 4.6. Let

$$\forall x,\ y \in X(x \prec y \Longrightarrow \exists z \in X(x \prec z \prec y))\ .$$

Then for any x $x \models \Box_\prec \Box_\prec A \longrightarrow \Box_\prec A,\ x \models \Box^*_\prec \Box^*_\prec A \longrightarrow \Box^*_\prec A$.

Proposition 4.7. If X satisfies the condition 2) and 3) of the definition of A-space, then for any $x \in X$

$$x \models \Box^*_\prec A \longrightarrow \Diamond^*_\leq(\beta\ \&\ \Box^*_\leq A).$$

Proposition 4.8. If X satisfies the condition 4) of the definition of A-space then for any x:

$$x \models \Box^*_\prec A \longrightarrow \Box^*_\prec \Diamond_\prec(\beta \& \Box^*_\leq A).$$

We finally establish the main result of this section.

Theorem about correctness .

1. In any A-space the following formulas are valid:

1) $\Box_{\leq} A \longrightarrow \Box_{\prec} A$;
1′) $\Box^*_{\leq} A \longrightarrow \Box^*_{\prec} A$;
2) $\Box_{\leq} A \longrightarrow \Box_{\leq}\Box_{\leq} A$;
2′) $\Box^*_{\leq} A \longrightarrow \Box^*_{\leq}\Box^*_{\leq} A$;
3) $\Box_{\leq} A \longrightarrow A$;
3′) $\Box^*_{\leq} A \longrightarrow A$;
4) $\Box_{\prec} A \longrightarrow \Box_{\prec}\Box_{\prec} A$;
4′) $\Box^*_{\prec} A \longrightarrow \Box^*_{\prec}\Box^*_{\prec} A$;
5) $\Box_{\prec} A \longrightarrow \Box_{\leq}\Box_{\prec} A$;
5′) $\Box^*_{\prec} A \longrightarrow \Box^*_{\leq}\Box^*_{\prec} A$;
6) $\Box_{\prec} A \longrightarrow \Box_{\prec}\Box_{\leq} A$;
6′) $\Box^*_{\prec} A \longrightarrow \Box^*_{\prec}\Box^*_{\leq} A$;
7) $\Box_{\prec} A \longrightarrow \Diamond_{\leq}(\beta \ \& \ \Box_{\leq} A)$;
7′) $\Box^*_{\prec} A \longrightarrow \Diamond^*_{\leq}(\beta \ \& \ \Box^*_{\leq} A)$;
8) $\Box_{\prec} A \longrightarrow \Box_{\prec}\Diamond^*_{\prec}(\beta \ \& \ \Box_{\leq} A)$;
8′) $\Box^*_{\prec} A \longrightarrow \Box^*_{\prec}\Diamond_{\prec}(\beta \ \& \ \Box^*_{\leq} A)$.

2. In A-space, where for all x, y there exists $x \cup y$, the following formula is valid:

$\Diamond_{\prec}\Box_{\prec} A \& \Diamond_{\prec}\Box_{\prec} B \longrightarrow \Diamond_{\prec}\Box_{\prec}(A \& B)$.

3. In A-space, if the relation $\prec$ is dense then the following formulas are valid:

$\Box_{\prec}\Box_{\prec} A \longrightarrow \Box_{\prec} A$ and $\Box^*_{\prec}\Box^*_{\prec} A \longrightarrow \Box^*_{\prec} A$.

5. A STRUCTURE OF A CANONICAL MODEL

Define the logical calculus L_0 by sets of axioms and derivation rules. The language L_0 includes the modalities described above.

The axioms of logic L_0 are:

1) All tautologies ;
2) $\Box_{\leq}(A \longrightarrow B) \ \& \ \Box_{\leq} A \longrightarrow \Box_{\leq} B$;
2′) $\Box^*_{\leq}(A \longrightarrow B) \ \& \ \Box^*_{\leq} A \longrightarrow \Box^*_{\leq} B$;
3) $\Box_{\prec}(A \longrightarrow B) \ \& \ \Box_{\prec} A \longrightarrow \Box_{\prec} B$;
3′) $\Box^*_{\prec}(A \longrightarrow B) \ \& \ \Box^*_{\prec} A \longrightarrow \Box^*_{\prec} B$;
4) $A \longrightarrow \Box_{\leq}\Diamond^*_{\leq} A$;
4′) $A \longrightarrow \Box^*_{\leq}\Diamond_{\leq} A$;
5) $A \longrightarrow \Box_{\prec}\Diamond^*_{\prec} A$;

5′) $A \longrightarrow \Box^*_{\prec}\Diamond_{\prec}A$.

The derivation rules of L_0 are:

1) $\frac{A}{\Box_{\leq}A}$;

1′) $\frac{A}{\Box^*_{\leq}A}$;

2) $\frac{A}{\Box_{\prec}A}$;

2′) $\frac{A}{\Box^*_{\prec}A}$;

3) $\frac{A,\ A\to B}{B}$.

Let L be a calculus containing all axioms and rules of the calculus L_0.

Denote by Φ_L the set of formulas of the logic L.

A finite sequence of formulas $A_1, \ldots,\ A_k \in \Phi_L$ is said to be a derivation in the logic L, if for all i A_i is an axiom or the immediate consequence of previous formulas by one of rules.

For any $A \in \Phi_l$, $\Gamma \subseteq \Phi_L$ let $\Gamma \longrightarrow_L A$ denote that there exists $k \geq 0$ and $A_1, \ldots,\ A_k \in \Gamma$, such that $(A_1 \longrightarrow \ldots (A_k \longrightarrow A))\ldots)$ is derived in L .

A set Γ of formulas is said to be L-contradictory iff there exists $B \in \Phi_L$ such that $\Gamma \longrightarrow_L B$ and $\Gamma \longrightarrow_L \neg B$. Otherwise, Γ is a L-noncontradictory set.

A set Γ is said to be L-complete set iff for all $A \in \Phi_L$ $A \in \Gamma$ or $\neg A \in \Gamma$.

A set Γ is said to be a L-theory iff Γ is closed under $\longrightarrow_L$.

Let W_L be a set of complete L-noncontradictory theories.

Define $\leq$ on W_L by $T_1 \leq T_2 \Longleftrightarrow \{A :\ \Box_{\leq}A \in T_1\} \subseteq T_2$.

Define $\prec$ on W_L similarly, i. e. $T_1 \prec T_2 \Longleftrightarrow \{A :\ \Box_{\prec}A \in T_1\} \subseteq T_2$.

If $A \in \Phi_L$ then let $U_A = \{T \in W_L :\ \Box_{\prec}A \in T\}$.

Let us introduce a topology τ on W_L, having taken B_τ as a basis, where

$$B_\tau = \{\emptyset\} \cup \{U_A :\ A \in \Phi_L\}\ .$$

Let $W^0_L = \{T \in W_L :\ \beta \in T\}$.

Thus we have the frame $\langle W_L,\ W^0_L,\ \leq,\ \prec\rangle$. The monadic predicate β is defined by the set W^0_L .

For all $T \in W_L$ let us introduce the notation $T^* = \{Q \in W_L :\ T \prec Q\}$.

Denote by $B_{\tau_0} = \{\emptyset\} \cup \{T^*_0 :\ T_0 \in W^0_L\}$.

Let us consider the model $M_L = \langle W_L,\ \leq,\ \prec,\ W^0_L,\ \models\rangle$, where we assume $T \models p \Longleftrightarrow p \in T$ for any variable p and $T \in W_L$. We shall call M_L the canonical model.

Lemma. For all $T,\ S \in W_L$

$$T\ R\ S \Longleftrightarrow \{A :\ \Box^*_R A \in S\} \subseteq T\ ,$$

where $R \in \{\leq, \prec\}$ (See [9]).

The following Lemma is proved by induction using the previous Lemma [9].

Lemma (about the canonical model). For all $A \in \Phi_L$, for all $T \in W_L$

$$T \models A \Longleftrightarrow A \in T \ .$$

Proposition 5.1. If $\Diamond_{\leq}\Diamond_{\prec}A \longrightarrow \Diamond_{\prec}A$ is derived in L then for all $T_1,\ T_2,\ T_3 \in W_L$

$$T_1 \leq T_2 \prec T_3 \Longrightarrow T_1 \prec T_3 \ .$$

Proposition 5.2. If $\Diamond_{\prec}\Diamond_{\leq}A \longrightarrow \Diamond_{\prec}A$ is derived in L, then for all $T_1,\ T_2,\ T_3 \in W_L$

$$T_1 \prec T_2 \leq T_3 \Longrightarrow T_1 \prec T_3 \ .$$

Proposition 5.3. If $\Box_{\leq}A \longrightarrow \Box_{\prec}A$ is derived in L then for all $T_1,\ T_2 \in W_L$

$$T_1 \prec T_2 \Longrightarrow T_1 \leq T_2 \ .$$

Proposition 5.4. If $(\Box_{\leq}A \longrightarrow A)\ ,\ (\Box_{\leq}A \longrightarrow \Box_{\leq}\Box_{\leq}A)\ ,$ $(\Box_{\prec}A \longrightarrow \Box_{\prec}\Box_{\prec}A)$ are derived in L, then the relation $\leq$ is the reflexive and the transitive relation, the relation $\prec$ is the transitive one (See [4]).

Proposition 5.5. If the axiom

$$\Box_{\prec}A \longrightarrow \Diamond_{\leq}(\beta \ \& \ \Box_{\prec}A)$$

is derived in L and the relation $\prec$ is transitive, then the set $B_{\tau_0} = \{\emptyset\} \cup \{T_0^* :\ T_0 \in W_L^0\}$ satisfies the following condition:

$$\forall U \in \tau \exists V \in B_{\tau_0}(V \subseteq U) \ .$$

Let L' be a calculus obtained by adding to L_0 the following axioms:

1) $\Box_{\leq}A \longrightarrow \Box_{\prec}A$;
1′) $\Box^*_{\leq}A \longrightarrow \Box^*_{\prec}A$;
2) $\Box_{\leq}A \longrightarrow \Box_{\leq}\Box_{\leq}A$;
2′) $\Box^*_{\leq}A \longrightarrow \Box^*_{\leq}\Box^*_{\leq}A$;
3) $\Box_{\leq}A \longrightarrow A$;
3′) $\Box^*_{\leq}A \longrightarrow A$;
4) $\Box_{\prec}A \longrightarrow \Box_{\prec}\Box_{\prec}A$;
4′) $\Box^*_{\prec}A \longrightarrow \Box^*_{\prec}\Box^*_{\prec}A$;
5) $\Box_{\prec}A \longrightarrow \Box_{\leq}\Box_{\prec}A$;
5′) $\Box^*_{\prec}A \longrightarrow \Box^*_{\leq}\Box^*_{\prec}A$;

6) $\Box_{\prec} A \longrightarrow \Box_{\prec}\Box_{\leq} A$;
6′) $\Box^*_{\prec} A \longrightarrow \Box^*_{\prec}\Box^*_{\leq} A$;
7) $\Box_{\prec} A \longrightarrow \Diamond_{\leq}(\beta \mathbin{\&} \Box_{\leq} A)$;
7′) $\Box^*_{\prec} A \longrightarrow \Diamond^*_{\leq}(\beta \mathbin{\&} \Box^*_{\leq} A)$;

Proposition 5.6. For all $T_1,\ T_2 \in W_L$

$$T_1 \leq T_2 \Longrightarrow \forall U \in \tau(T_1 \in U \Longrightarrow T_2 \in U)\ .$$

We finally establish the main results of this section.

Theorem on the canonical model. The canonical model for L' possesses the following properties:

1) $T_1 \leq T_2 \prec T_3 \Longrightarrow T_1 \prec T_3$;
2) $T_1 \prec T_2 \leq T_3 \Longrightarrow T_1 \prec T_3$;
3) $T_1 \prec T_2 \Longrightarrow T_1 \leq T_2$;
4) $T \leq T$;
5) $T_1 \leq T_2 \leq T_3 \Longrightarrow T_1 \leq T_3$;
6) $T_1 \prec T_2 \prec T_3 \Longrightarrow T_1 \prec T_3$;
7) $\forall U \in \tau \exists V \in B_{\tau_0}(V \subseteq U)$;
8) $T_1 \leq T_2 \Longrightarrow \forall U \in \tau(T_1 \in U \longrightarrow T_2 \in U)$.

The following Theorem is obtained by standard reasoning [4] from Lemma and Theorem about the canonical model.

Theorem 5.7. If a formula A is valid in the frames, where the following conditions hold:

1) $T_1 \leq T_2 \prec T_3 \Longrightarrow T_1 \prec T_3$;
2) $T_1 \prec T_2 \leq T_3 \Longrightarrow T_1 \prec T_3$;
3) $T_1 \prec T_2 \Longrightarrow T_1 \leq T_2$;
4) $T \leq T$;
5) $T_1 \leq T_2 \leq T_3 \Longrightarrow T_1 \leq T_3$;
6) $T_1 \prec T_2 \prec T_3 \Longrightarrow T_1 \prec T_3$,

then it is derived in L'.

Examples.

Let us consider the partially ordered set $\langle X,\ \leq,\ \prec\rangle$ where $\langle X,\ \leq\rangle$ is a tree, i. e. $(x \leq y$ and $x \leq z \Longrightarrow (y \leq z$ or $z \leq y))$.
Assume $\prec\ =\ \leq$, and U is an open set if $\forall x(x \in U$ and $x \leq y \Longrightarrow y \in U)$.

Let $X_0 = X$, then $\langle X,\ X_0,\ \leq,\ \prec\rangle$ be A-space.

Now consider poset $\langle X,\ \geq,\ \succ\rangle$ where $\langle X,\ \geq\rangle$ is a tree, i. e. $(x \geq y$ and $x \geq z \Longrightarrow (y \geq z$ or $z \geq y))$.
Assume $\succ\ =\ \geq$, and U is an open set if $\forall x(x \in U\ \textit{and}\ x \geq y \Longrightarrow y \in U)$.

Let $X_0 = X$, then $\langle X,\ X_0,\ \geq,\ \succ\rangle$ will be A-space too.

Since S_4 is complete over tree — like models [4], then we obtain:

Theorem 5.8.

1) A formula containing modalities of form $\Box_{\leq}$ only is valid in all A-spaces iff it is derived in S_4.

2) A formula containing modalities of form $\Box^*_{\leq}$ only is valid in all A-spaces iff it is derived in S_4.

3) If a formula containing modalities of form $\Box_{\prec}$ only is valid in all A-spaces then it is derived in S_4.

4) If a formula containing modalities of form $\Box^*_{\prec}$ only is valid in all A-spaces then it is derived in S_4.

6. TOPOLOGICAL PROPERTIES OF THE CANONICAL MODEL

Let L be any extension of the logic L_0. Let us present two more interesting results.

Proposition 6.1. For the canonical model:

$$B_{\tau_0} = \{\emptyset\} \cup \{T_0^* : \ T_0 \in W_L^0\} \text{ is a basis} \Longleftrightarrow$$

$$\forall A \in \Phi_L \forall T \in W_L (\Box_{\prec} A \in T \Longrightarrow \Diamond^*_{\prec}(\beta \& \Box_{\prec} A) \in T).$$

Proposition 6.2. If the formulas $\Box_{\prec}\Box_{\prec} A \longrightarrow \Box_{\prec} A$, $\Box_{\prec} A \longrightarrow \Box_{\prec}\Diamond^*_{\prec}(\beta \ \& \ \Box_{\prec} A)$ are derived in L, and the set $\{\emptyset\} \cup \{T_0^* : \ T_0 \in W_L^0\}$ is a basis of topology τ, then

$$\forall U \in \tau \forall T \in U \exists T_0 \in W_L^0 \cap U (T_0 \prec T).$$

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ПОЛИМОДАЛЬНАЯ ЛОГИКА НА ОСНОВЕ A-ПРОСТРАНСТВ

Препринт
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