Lemma 1. Given $E \in \mathbb{E}_L^b$ and $X \in \text{Conf}(E)$, (i) $E \setminus X \in \mathbb{E}_L^b$; (ii) $X \cup X'$ is conflict-free and $\mathbb{z}(X \cup X') = \mathbb{z}(X) \cup \mathbb{z}'(X')$, for any $X' \in \text{Conf}(E \setminus X)$.

Proof. (i) It follows from the definitions of the components of $E'$.

(ii) We show that $X \cup X'$ is conflict-free. Suppose a contrary, i.e. there is $e' \in X'$ such that $e' \in \mathbb{z}(X)$. Then, due to the definition of $\rightarrow'$, $e' \rightarrow e'$, contradicting $e' \in X' \in \text{Conf}(E')$.

We know that $\mathbb{z}(X \cup X') = \{ \bar{x} \in E \mid \bar{x} \not\in \mathbb{x} \}$. Then, $\mathbb{z}(X \cup X') = \mathbb{z}(X) = \{ \bar{x} \in E \mid \bar{x} \not\in X \} \cup A = \{ \bar{x} \in E \mid \bar{x} \not\in X \} \cup \{ \bar{x} \in E \mid \bar{x} \not\in X' \}$. On the other hand, we have that $\mathbb{z}'(X') = \{ \bar{x} \in E \mid \bar{x} \not\in X' \}$. Then, $\mathbb{z}'(X') = \{ \bar{x} \in E \mid \bar{x} \not\in X' \}$, by the definition of $\mathbb{z}'$. As $X \cup X'$ is conflict-free, $\mathbb{z}'(X') = \{ \bar{x} \in E \mid \bar{x} \not\in X' \} = A$.

Lemma 2. Given, $E \in \mathbb{E}_L^b$ and $X \in \text{Conf}(E)$, (i) $E \setminus X \in \mathbb{E}_L^b$; (ii) $X \cup X'$ is conflict-free and $\mathbb{z}(X \cup X') = \mathbb{z}(X) \cup \mathbb{z}'(X')$, for any $X' \in \text{Conf}(E \setminus X)$.

Proof. (i) It follows from the definitions of the components of $E \setminus X$.

(ii) We show that $X \cup X'$ is conflict-free. Suppose a contrary, i.e. there is $e' \in X'$ such that $e' \in \mathbb{z}(X)$. Then, $(\emptyset, e') \rightarrow e'$, contradicting $e' \in X' \in \text{Conf}(E')$.

The proof of the fact that $\mathbb{z}(X \cup X') = \mathbb{z}(X) \cup \mathbb{z}'(X')$ is similar to that of the same fact in the previous lemma.

Lemma 3. Given $E \in \mathbb{E}_L^b$ and $X \in \text{Conf}(E)$,

(i) $E \setminus X \in \mathbb{E}_L^b$;

(ii) for any $X' \in \text{Conf}(E \setminus X),

(a) whenever $b \not\prec a'$: (*) if $a \in X \cup X'$ and $b \in X'$, then $a \not\prec X'$; (**) if $a \in X$ and $b \in X \cup X'$, then $b \not\prec X'$; (***) if $a, b \in X'$, then $b \not\prec a$;

(b) $X \cup X'$ is conflict-free and $\mathbb{z}(X \cup X') = \mathbb{z}(X) \cup \mathbb{z}'(X')$.

Proof. (i) It follows from the definitions of the components of $E \setminus X$.

(ii) Take an arbitrary $X' \in \text{Conf}(E \setminus X)$.

(a) Assume $b \not\prec a$.

Suppose a contrary, i.e. $a \not\in X'$. Then, $a \in X$. As $X$ is left-closed up to conflicts, there is $y \in X$ such that $b \not\in y \not\prec a$. Hence, $b \in \mathbb{z}(X)$. By the definition of $\mathbb{z}'$, $b \not\prec b$, contradicting $b \in X'$. Thus, $a \in X'$.

Suppose a contrary, i.e. $b \not\in X$. As $X$ is left-closed up to conflicts, there is $y \in X$ such that $b \not\in y \not\prec a$. Then, $b \in \mathbb{z}(X)$. By the definition of $\mathbb{z}'$, $b \not\prec b$, contradicting $b \in X'$. Thus, $b \in X$.

Suppose a contrary, i.e. $\neg(b \not\prec a)$. Due to the definition of $\not\prec'$, we get that $(b, a) \in \not\prec(X)$, i.e. $b \not\prec y \not\prec a$, for some $y \in X$. By the definition of $\mathbb{z}'$, $b \not\prec b$, contradicting $b \in X'$. Thus, $b \not\prec a$.

(b) We show that $X \cup X'$ is a conflict-free set. Suppose a contrary, i.e. there is $e' \in X'$ such that $e' \not\in \mathbb{z}(X)$. Then, $e' \not\prec e'$, contradicting $e' \in X'$.

We know that $\mathbb{z}(X \cup X') = \{ \bar{x} \in E \mid \bar{x} \not\in \mathbb{x} \}$. Then, $\mathbb{z}(X \cup X') = \mathbb{z}(X) = \{ \bar{x} \in E \mid \bar{x} \not\in X \} \cup A = \{ \bar{x} \in E \mid \bar{x} \not\in X \} \cup \{ \bar{x} \in E \mid \bar{x} \not\in X' \}$. On the other hand, we have that $\mathbb{z}'(X') = \{ \bar{x} \in E \mid \bar{x} \not\in X' \}$. Then, $\mathbb{z}'(X') = \{ \bar{x} \in E \mid \bar{x} \not\in X' \}$, by the definition of $\mathbb{z}'$. Due to $X \cup X'$ being conflict-free, $\mathbb{z}'(X') = \{ \bar{x} \in E \mid \bar{x} \not\in X' \} = A$. 
Lemma 4. Given $E \in \mathbb{E}_{d}^{s/g}$, $X \in \text{Conf}(\mathcal{E})$, and $\mathcal{E}' = \mathcal{E} \setminus X$,

(i) $A \cap E' \in \text{Con}'$, for all $A \in \text{Con}$, and $A' \in \text{Con}$, for all $A' \in \text{Con}'$;
(ii) $\mathcal{E}' \in \mathbb{E}_{d}^{s/g}$;
(iii) $X \cup X'$ is conflict-free and $\mathcal{z}(X \cup X') = \mathcal{z}(X) \cup \mathcal{z}(X')$, for any $X' \in \text{Conf}(\mathcal{E}')$.

Proof. (i) Take an arbitrary $A \in \text{Con}$. This implies that $A \subseteq E$ and $\neg(x \not\in X')$, for all $x,x' \in A$. Check that $C \cap E' \in \text{Con}'$. Suppose a contrary, i.e. there is $a,a' \in A \cap E'$ such that $a \not\in a'$. By the definition of $\mathcal{z}'$, this means that $a,a' \in A$ and $a \not\in a'$, contradicting $A \in \text{Con}$. So, $A \cap E' \in \text{Con}'$. Next, take an arbitrary $A' \in \text{Con}'$. We shall show that $A' \in \text{Con}$. Assume a contrary, i.e. there is $a,a' \in A'$ such that $a \not\in a'$. Since $A' \in \text{Con}'$, it holds that $A' \subseteq E'$. By the definition of $\mathcal{z}'$, we get $a \not\in a'$, contradicting $A' \in \text{Con}'$. Hence, $A' \in \text{Con}$.

(ii) Due to the definitions of the components of $\mathcal{E} \setminus X$, it is a $G$-structure over $L$. We shall show that $A' \vdash e, B' \vdash e$, and $\text{Con}'(A' \cup B' \cup \{e\}) \Rightarrow A' \cap B' \vdash e$, if $\mathcal{E}$ is an $S$-structure over $L$. Suppose that $A' \vdash e, B' \vdash e$, and $\text{Con}'(A' \cup B' \cup \{e\})$. Then, for some $A' \subseteq A'$ and $B' \subseteq B'$, we get $(A',e) \in \vdash_{\text{min}}$, and $(B',e) \in \vdash_{\text{min}}$, and, moreover, $\text{Con}'(A' \cup B' \cup \{e\})$. This implies that there is $(A,e) \in \vdash_{\text{min}}$, and $(B,e) \in \vdash_{\text{min}}$ such that $A' = A \cap E'$, $B' = B \cap E'$, $\{e\} \cup X \in \text{Con}'$. Hence, $(A \cap B) \cup \{e\} \in \vdash_{\text{min}}$, due to $\mathcal{E}$ being a stable event structure. By the definition of $\vdash_{\text{min}}$, $(A \cap B) \cup \{e\} \in \vdash_{\text{min}}$.

(iii) Next, we show that $X \cup X' \in \text{Con}$. As $X \in \text{Conf}(\mathcal{E})$, $(X' \in \text{Conf}(\mathcal{E}'))$, we get $X \in \text{Con}$, $(X' \in \text{Con}')$. By item (i), it holds that $X' \in \text{Con}$. Suppose a contrary, i.e. $X \cup X' \not\in \text{Con}$. Then, we can find $e' \in X'$ and $e \in X$ such that $e' \not\in e$. Since $X' \in \text{Conf}(\mathcal{E}')$, there are $e'_1, \ldots, e'_m (m \geq 0)$ such that $X' = \{e'_1, \ldots, e'_m\}$ and $\vdash_{\text{min}} = \vdash_{\text{min}} - \{e'_j\}$, for all $i < m$. W.l.o.g., assume $e' = e_j$ for some $1 \leq j \leq m$. By the definition of $\vdash_{\text{min}}$, there is $W' \subseteq \{e'_1, \ldots, e'_{j-1}\}$ such that $(W',e_j) \in \vdash_{\text{min}}$. Then, due to the definition of $\vdash_{\text{min}}$, there is $(W',e_j) \in \vdash_{\text{min}}$ such that $W' \cap E' \cap \{e'_j\} \cup X \in \text{Conf}'$, contradicting $e' \not\in e \in E$.

Check that $W \cup X \cup X' \in \text{Conf}$ iff $(W \setminus X) \cup X' \in \text{Con}'$ and $W \cup X \in \text{Con}$. Assume $W \cup X \cup X' \in \text{Con}$. Since $W \cup X \subseteq W \cup X \cup X'$, it holds $W \cup X \in \text{Con}$. Next, it easy to see that $(W \cup X \cup X') \cap E' \in \text{Con}'$, due to item (i). This means $(W \setminus X) \cup X' \in \text{Con}'$.

Conversely, suppose $(W \setminus X) \cup X' \in \text{Con}'$ and $W \cup X \in \text{Con}$. By item (i), we have $(W \setminus X) \cup X' \in \text{Con}$. Assume that $W \cup X \cup X' \not\in \text{Con}$. Hence, there are $a,b \in W \cup X \cup X'$ such that $a \not\in b$. Consider possible cases:

- $a \in W \cup X$. Since $W \cup X \in \text{Con}$, $b \not\in W \cup X$. So, $b \in X'$. As $(W \setminus X) \cup X' \in \text{Con}'$ and $a \not\in b$, we have $a \in X$, contradicting $X \cup X' \in \text{Con}$.
- $a \in X'$. Since $(W \setminus X) \cup X' \in \text{Con}$, we get $b \in X$, contradicting $X \cup X' \in \text{Con}$.

Thus, $W \cup X \cup X' \in \text{Con}$.
Lemma 5. Given $E \in \mathbb{E}_L^c$ and $X \in C$, $E \setminus X \in \mathbb{E}_L^c$.

Proof. It follows from the definitions of the components of $E \setminus X$.

Lemma 6. (i) An RC-structure $E = (E, \vdash, L, l)$ over $L$ can be transformed into $\tilde{E} = (E, \tilde{C}, L, l)$ in $\mathbb{E}_L^c$ such that $LC(E) = LC(\tilde{E})$; and, moreover, $\text{Conf}(E) = \text{Conf}(\tilde{E})$, if $E$ is a pure RC-structure;

(ii) $E(\tilde{E}) = \tilde{E}$, for $\tilde{E} \in \mathbb{E}_L^c$;

(iii) $C(E(C)) = C$, for $C \in \mathbb{E}_L^c$.

Proof. (i) For the transformation, we can use the bijective mappings $C$ and $E$ from [13] defined as follows: $C(E) = (E, LC(E), L, l)$ and $\tilde{E} = E(C(E)) = (E, \tilde{C}, L, l)$, with $A \vdash B$ iff $A \cap B = \emptyset$ and $A \cup B \in LC(E)$. It is easy to see that $\tilde{E} \in \mathbb{E}_L^c$. The fact that $LC(E) = LC(\tilde{E})$ follows from Theorem 2 [13].

Assume $E$ is a pure RC-structure and $* = \text{step}$. Take an arbitrary $X \in \text{Conf}_*(E)$. W.l.o.g. suppose that $\emptyset = X_0 \rightarrow \ldots X_1 \rightarrow \ldots X_n = X (n \geq 0)$ in $E$.

Clearly, $X_i \in LC(E)$, for all $0 \leq i \leq n$. So, we have that $Y \subseteq LC(E)$, for all $X_i \subseteq Y \subseteq X_{i+1}$ and $0 \leq i < n$. Then, we get that $Y \subseteq LC(\tilde{E})$, for all $X_i \subseteq Y \subseteq X_{i+1}$ and $0 \leq i < n$. Hence, it holds that $X_i \rightarrow X_{i+1}$ in $\tilde{E}$. Hence, $X = X_n \in \text{Conf}_*(E)$.

Take an arbitrary $X \in \text{Conf}_*(\tilde{E})$. Applying analogous reasonings as in the proof of the opposite direction, we obtain that $X \in \text{Conf}_*(E)$.

(ii) Take an arbitrary $E = (E, \vdash, L, l) \in \mathbb{E}_L^c$, with $\vdash = \{(A, B) \mid B \subseteq C \subseteq LC(E), A = C \setminus B\}$. We need to show that $E(C(E)) = \tilde{E}$. It is known that $C(E) = (E, LC(E), L, l)$ and $E(C(E)) = (E, \tilde{C}, L, l)$, with $\tilde{C} = \{(A, B) \mid A \cap B = \emptyset, A \cup B \in LC(E)\}$. It is easy to see that $\vdash = \tilde{\vdash}$.

(iii) Take an arbitrary $C = (E, C, L, l) \in \mathbb{E}_L^c$. We have to check that $C(E(C)) = C$. By definition, $E(C) = (E, \vdash, L, l)$, with $\vdash = \{(A, B) \mid A \cap B = \emptyset, A \cup B \in C\}$. and $E(C(E)) = (E, LC(E(C)), L, l)$. We need to show that $C = LC(E(C))$.

Consider an arbitrary $A' \subseteq A \subseteq C$. Define $B' = A \setminus A'$. Then, $A' \cap B' = \emptyset$ and $A' \cup B' = A \in C$, i.e. there is $B' \subseteq A$ such that $B' \vdash A'$. Hence, $A \in LC(E(C))$.

Assume $A \in LC(E(C))$. By definition, for all $A' \subseteq A$ there is $A'' \subseteq A$ s.t. $A'' \vdash A'$. Due to the definition of $\vdash$, we get $A' \cap A'' = \emptyset$ and $A' \cup A'' \in C$. Take $A'' = A$. Then $A \cup A'' = A \in C$.

Lemma 7. Let $E \in \mathbb{E}_L^c$ with $X \in LC(E)$ and $C \in \mathbb{E}_L^c$ with $Y \in C$. Then,

(i) $E \setminus X \in \mathbb{E}_L^c$;

(ii) $(X \cup Z) \in LC(E)$, for any $Z \subseteq E$ $\iff$ $Z \subseteq LC(E \setminus X)$;

(iii) $E \setminus X$ is in the standard form;

(iv) $C(E \setminus X) = C(E) \setminus X$ and $E(C(E \setminus X)) = E \setminus X$;

(v) $E(C \setminus Y) = E(C) \setminus Y$ and $E(C \setminus Y) = C \setminus Y$.

Proof. (i) This follows from the definitions of the components of $E \setminus X$.

(ii) $(\Rightarrow)$ First, notice that $X \cap Z = \emptyset$. Suppose $(X \cup Z) \in LC(E)$. Then, for all $Z \subseteq Z$, $(Z \cup \tilde{Z}) \in LC(E)$, where $\tilde{Z} = X \cup Z \setminus \tilde{Z}$. As $E$ is in the standard form, $Z \vdash \tilde{Z}$, for all $Z \subseteq Z$ and the corresponding $\tilde{Z}$. Obviously, $(Z \cup (\tilde{Z}' = ...
\[ \tilde{Z} \setminus X \cup X \in LC(E) \text{ and } \tilde{Z}' \subseteq Z. \text{ Due to the definition of } \vdash', \text{ for all } \tilde{Z} \subseteq Z, \text{ there exists } \tilde{Z}' \subseteq Z \text{ such that } \tilde{Z} \vdash' \tilde{Z}. \text{ Thus, } Z \in LC(E \setminus X). \]

\((\Leftarrow)\) Assume \(Z \in LC(E \setminus X)\). Then, for all \(\tilde{Z} \subseteq Z\), there is \(\tilde{Z} \subseteq Z\) such that \(\tilde{Z} \vdash' \tilde{Z}\). Then, \(\emptyset \vdash' Z\), due to item (i). By the definition of \(\vdash'\), this implies that \((X \cup Z) \in LC(E)\).

\((\Rightarrow)\) Suppose \(A' \vdash' B'\). Then, \(A' = (A' \cup B') \setminus B'\), by item (i). Moreover, \((A' \cup B' \cup X) \in LC(E)\), due to the definition of \(\vdash'\). Thanks to item (ii), we get that \((A' \cup B') \in LC(E \setminus X)\).

\[(\Leftarrow)\) Assume \(C' \in LC(E \setminus X)\). Take \(B' \subseteq C'\) and \(A' = C' \setminus B'\). According to item (ii), \((C' \cap X) = (A' \cup B' \cup X) \in LC(E)\). Moreover, as \((A' \cup X) \cap B' = \emptyset\), \(A' \cup X \vdash' B'\), due to \(E\) being in the standard form. Hence, \(A' \vdash' B'\), by the definition of \(\vdash'\).

(iv) We know that \(E \setminus X = (E \setminus X, \vdash', L, l \mid_{E \setminus X})\), where \(A \vdash B\) iff there is \(A \vdash B\), \(A = A \cap (E \setminus X)\), \(B = B \cap (E \setminus X)\), and \(A' \cup B' \cup X \in LC(E)\). Hence, \(C(E \setminus X) = (E \setminus X, L, l \mid_{E \setminus X})\). On the other hand, \(C(E \setminus X) = (E, LC(E), L, l)\setminus X = (E \setminus X, M = \{B \subseteq E \setminus X \mid B \cup Y \in LC(E)\}, L, l \mid_{E \setminus X})\).

Due to item (ii), we get that \(LC(E \setminus X) = M\). As \(E \setminus X \in E_X\) by item (i), we may conclude that \(E(E \setminus X) = E \setminus X\), by item (ii) of Lemma 6.

(v) We know that \(E(C) = (E, \vdash, L, l)\), with \(A \vdash B\) iff \(A \cap B = \emptyset\) and \(A \cup B \in C\), and \(E(C) \setminus Y = (E, B \vdash' L, l \mid_{E \setminus Y})\), where \(A \vdash B\) iff there is \(A' \cup B'\) s.t. \(A = A' \cap (E \setminus Y)\), \(B = B' \cap (E \setminus Y)\), and \(A \cup B \cap Y \in LC(E(C))\). On the other hand, by definition, \(C \setminus Y = (E \setminus Y, M = \{B \subseteq E \setminus Y \mid B \cup Y \in C\}, L, l \mid_{E \setminus Y})\).

Then, \(E(C \setminus Y) = (E \setminus Y, \vdash', L, l \mid_{E \setminus Y})\), where \(A \vdash' B\) iff \(A \cap B = \emptyset\) and \(A \cup B \in M\). It is sufficient to show that \(A \vdash' B\) iff \(A \vdash'' B\). Assume \(A \vdash' B\). This means that there is \(A' \vdash' B'\) such that \(A = A' \cap (E \setminus Y)\), \(B = B' \cap (E \setminus Y)\), and \(A \cup B \cap Y \in LC(E(C))\). Since \(A' \vdash' B'\), we have \(A' \cap B' = \emptyset\) and \(A' \cup B' \in C\). So, \(A \cap B = \emptyset\). Due to \(A \cup B \cap Y \in LC(E(C))\), it holds for all \(Z \subseteq A \cup B \cap Y\) there is \(Z' \subseteq A \cup B \cap Y\) s.t. \(Z' \vdash Z\), i.e. \(Z' \cap Z = \emptyset\) and \(Z \cup Z' \in C\). Take \(Z = A \cup B \cap Y\). Then, \(Z' = \emptyset\) and \(Z \cup Z' = A \cup B \cap Y \in C\).

This means that \(A \cup B \cap M\) and \(A \cap B = \emptyset\). Hence, \(A \vdash'' B\). Next, assume \(A \vdash'' B\). By the definition of \(\vdash''\), we have \(A \cup B \in M\) and \(A \cap B = \emptyset\). Hence, \(A \cup B \cap Y \in C\). Due to the definition of \(\vdash\), we may conclude that \(A \cup B \cap Y \in C\). Check that \(A \cup B \cap Y \in LC(E(C))\). Take an arbitrary \(Z \subseteq A \cup B \cap Y\). Let \(Z' = (A \cup B \cap Y) \setminus Z\). Then, \(Z \cup Z' = A \cup B \cap Y \in C\) and \(Z \cap Z' = \emptyset\). Hence, \(Z' \vdash Z\). Thus, \(A \cup B \cap Y \in LC(E(C))\). By the definition of \(\vdash'\), we get \(A \vdash' B\). Thanks to item (iii) of Lemma 6, we have \(C(E(C \setminus Y)) = S(C \setminus Y)\).

**Proposition 1.** Let \(E\) be a structure over \(L\). Then,

(i) for any \(E' = E \setminus X\), with \(X \in Conf_*(E)\), and \(E'' = E' \setminus X'\), with \(X' \in Conf_*(E')\), \(X \cup X' \in Conf_*(E')\) and \(E'' = E \setminus (X \cup X')\);

(ii) for any \(X, X'' \in Conf_*(E)\) such that \(X \rightarrow X''\), \(X'' \setminus X \in Conf_*(E \setminus X)\) and, moreover, \(\emptyset \rightarrow X''\), \(X'' \setminus X \in \mathcal{E}(E \setminus X)\).

**Case with \(E \in \mathcal{E}_{lp}^{sp}\) and \(* = pom.**

**Proof.** (i) Let \(E' = E \setminus X\), with \(X \in Conf_*(E)\), and \(E'' = E' \setminus X'\), with \(X' \in Conf_*(E')\).
Check that $X \cup X' \in \text{Conf}_{*}(E)$. Since $X \in \text{Conf}_{*}(E) \ (X' \in \text{Conf}_{*}(E'))$, $X \ (X')$ is finite, conflict-free, left-closed, does not contain enabling cycles, and $\emptyset \rightarrow_{*}^{X} X \in E \ (\emptyset \rightarrow_{*}^{X'} X' \in E')$.

First, it is clear that $X \cup X'$ is a finite subset of events of $E$, by the definition of $E'$.

Second, $X \cup X'$ is conflict-free, due to Lemma 1 (ii).

Third, we verify that $X \cup X'$ is left-closed. Suppose that $e \in X \cup X'$ and $d \rightarrow e$ for some $d \in E$. We have to show that $d \in X \cup X'$. The case when $e = d$ is obvious. Assume $e \neq d$. If $e \in X$, then the result follows from the left-closedness of $X$. Consider the case when $e \in X'$. If $e \notin \sharp(X)$, then, due to the definition of $\rightarrow'$, $e \rightarrow' e$, contradicting the fact that $X'$ does not contain enabling cycles. Hence, $e \notin \sharp(X)$. Suppose a contrary, i.e. $d \notin X \cup X'$. Then, $d \in E'$, because $d \notin X$. By the definition of $\rightarrow'$, $d \rightarrow' e$, contradicting the left-closedness of $X'$.

Fourth, we need to show that $X \cup X'$ does not contain enabling cycles. Assume a contrary, i.e. there are $x_{1}, \ldots, x_{n} \in X \cup X'$ such that $x_{1} \rightarrow \ldots \rightarrow x_{n} \rightarrow x_{1}$ for some $n \geq 1$. Consider two possible cases:

$x_{1} \in X$ Since $X$ is left-closed, we have that $x_{n}, \ldots, x_{2}, x_{1} \in X$, contradicting the fact that $X$ does not contain enabling cycles.

$x_{1} \in X'$ Assume that there is $k \leq n$ such that $x_{k} \in X$. Then, due to the fact that $X$ is left-closed, we get $x_{k-1}, \ldots, x_{1} \in X$, contradicting $x_{1} \in X'$. Hence, $x_{1}, \ldots, x_{n} \in X'$. This implies, $x_{1} \rightarrow' \ldots \rightarrow' x_{n} \rightarrow' x_{1}$, by the definition of $\rightarrow'$, contradicting the fact that $X'$ does not contain enabling cycles.

Thus, $X \cup X'$ does not contain enabling cycles.

So, $X \cup X'$ is a configuration of $E$. Clearly, $\emptyset \rightarrow_{*} X \cup X'$. Thus, $X \cup X'$ is a configuration in $*$-semantics of $E$.

We shall prove that $E' \setminus X' = E'' = \bar{E} = E \setminus (X \cup X')$.

By definition, $E = E \setminus (X \cup X') = (E \setminus X) \setminus X' = E' \setminus X' = E''$.

Again by definition, $\sharp'' = \sharp \cap (E'' \times E'') = \sharp \cap (E' \times E') \cap (E'' \times E'') = \sharp \cap (E'' \times E'') = \sharp''$.

Consider arbitrary $a, b \in E''$ such that $a \rightarrow'' b$. Then, $a, b \notin (X \cup X')$. By the definition of $\rightarrow''$, two cases are possible:

- $b \notin \sharp'(X')$. Then, $a \rightarrow b$. Due to the definition of $\rightarrow'$, we have to check two cases. If $b \notin \sharp(X)$, then $a \rightarrow b$. By Lemma 1(ii), $b \notin \sharp(X \cup X')$. This implies that $a \rightarrow b$. If $b \in \sharp(X)$, then $a = b$ and, moreover, $b \in \sharp(X \cup X')$, due to Lemma 1(ii). Hence, $a \rightarrow b$.

- $b \in \sharp'(X')$. Then, $a = b$ and, moreover, $b \in \sharp(X \cup X')$, due to Lemma 1(ii). Hence, $a \rightarrow b$.

Conversely, consider arbitrary $a, b \in \bar{E}$ such that $a \rightarrow b$. Then, $a, b \notin (X \cup X')$. By the definition of $\rightarrow$, two cases are admissible:

- $b \notin \sharp(X \cup X')$. Then, $a \rightarrow b$. By Lemma 1(ii), $b \notin \sharp(X)$ and, moreover, $b \notin \sharp'(X')$. Hence, $a \rightarrow b$ and, moreover, $a \rightarrow'' b$.

- $b \in \sharp(X \cup X')$. Then, $a \rightarrow b$. Due to Lemma 1(ii), we have to consider two cases. If $b \in \sharp(X)$, then $b \rightarrow b$. As $b \in \bar{E} = E''$, we get $b \rightarrow'' b$. If $b \in \sharp'(X')$, then $b \rightarrow'' b$. 


Finally, we get that \( l'' = l \big|_{E''} = l \big|_{E} = \bar{l} \).

(ii) Let \( X \rightarrow_* X'' \) in \( \mathcal{E} \). Since \( X \) and \( X'' \) are configurations of \( \mathcal{E} \), \( X \) and \( X'' \) are finite subsets of \( E \), conflict-free, left-closed, and do not contain enabling cycles.

First, it is clear that \( X' = X'' \setminus X \) is a finite subset of \( E' \), by the definition of \( E' \).

Second, we show that \( X' \) is a conflict-free subset of \( E' \). Suppose a contrary, i.e. there is \( e, e' \in X' \) such that \( e \not\gamma e' \). Obviously, \( X' \) is a conflict-free subset of \( E \). Then, \( \neg(e \not\gamma e') \), contradicting the definition of \( \not\gamma \).

Third, we check that \( X' \) is left-closed. Assume that \( e \in X' \) and \( d \not\gamma X' \), for some \( d \in E' \). Clearly, \( d \neq e \). By virtue of the definition of \( \rightarrow_* \), two cases are admissible.

- \( e \not\gamma \bar{z}(X) \). Then, \( d \rightarrow e \). Due to the left-closedness of \( X'' \), we have that \( d \in X'' \), because \( e \in X'' \). As \( d \not\gamma X' \), we get \( d \in X' \), contradicting \( d \in E' \).

- \( e \in \bar{z}(X) \). As \( X \subseteq X'' \) and \( e \in X'' \), this contradicts the conflict-freeness of \( X'' \).

Fourth, it is straightforward to show that \( X' \) does not contain enabling cycles. Hence, \( X' \in \text{Conf}(\mathcal{E} \setminus X) \).

Clearly, \( \emptyset \rightarrow_* X' \) in \( \mathcal{E} \setminus X \). Thus, \( X' \) is a configuration in \( \ast \)-semantics of \( \mathcal{E} \setminus X \).

\[ \square \]

**Case with \( \mathcal{E} \in \mathcal{E}_{L}^{b/d} \) and \( \ast = \text{mset} \).**

**Proof.**

(i) Let \( \mathcal{E}' = \mathcal{E} \setminus X \), with \( X \in \text{Conf}(\mathcal{E}) \), and \( \mathcal{E}'' = \mathcal{E}' \setminus X' \), with \( X' \in \text{Conf}(\mathcal{E}') \).

Check that \( X \cup X' \in \text{Conf}(\mathcal{E}) \). Since \( X \in \text{Conf}(\mathcal{E}) \) \((X' \in \text{Conf}(\mathcal{E}')) \), \( X \) \((X') \) is a finite conflict-free, secured subset of \( E \) \((E') \), and \( \emptyset \rightarrow_* X \) in \( \mathcal{E} \) \((\emptyset \rightarrow_* X' \) in \( \mathcal{E}' \)).

Obviously, \( X \cup X' \) is a finite subset of events of \( E \), by the definition of \( E' \).

Next, we show that \( X \cup X' \) is conflict-free. Suppose a contrary, i.e. there is \( e' \in X' \) such that \( e' \in \bar{z}(X) \). Then, \( (\emptyset, e') \in \rightarrow_* \), contradicting \( e' \in X' \).

Finally, we check that \( X \cup X' \) is secured. As \( X \) is secured, there is a sequence \( e_1 \ldots e_n \) (\( n \geq 0 \)) such that \( X = \{e_1, \ldots, e_n\} \), and for all \( i < n \), if \( W \rightarrow e_{i+1} \), then \( \{e_1, \ldots, e_i\} \cap W \neq \emptyset \). Also, there exists a sequence \( e_{i} \ldots e_{m} \) (\( m \geq 0 \)) such that \( X' = \{e_1, \ldots, e_m\} \), and for all \( j < m \), if \( W \rightarrow e_{j+1} \), then \( \{e_1, \ldots, e_j\} \cap W \neq \emptyset \), because \( X' \) is secured. Consider a sequence \( e_1 \ldots e_n e_{n+1} = e_1 \ldots e_{n+m} = e_m \).

Clearly, \( \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}\} = X \cup X' \). Assume \( W \rightarrow e_{j+1} \) for some \( j < n + m \). If \( j < n \), we know that \( \{e_1, \ldots, e_j\} \cap W \neq \emptyset \). Consider the case when \( n \leq j < n + m \). If \( W \cap X \neq \emptyset \), then \( \{e_1, \ldots, e_j\} \cap W \neq \emptyset \). Otherwise, we get \( W \rightarrow e_{j+1} \), by the definition of \( \rightarrow_* \). Then, we know that \( \{e_{n+1}, \ldots, e_j\} \cap W \neq \emptyset \). So, \( W \cap \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_j\} \neq \emptyset \). Hence, \( X \cup X' \) is a configuration of \( \mathcal{E} \). Clearly, \( \emptyset \rightarrow_* X \cup X' \). Thus, \( X \cup X' \) is a configuration in \( \ast \)-semantics of \( \mathcal{E} \).
Proof. (i) Let $\mathcal{E}' = \mathcal{E} \setminus X$, with $X \in \text{Conf}_s(\mathcal{E})$, and $\mathcal{E}'' = \mathcal{E}' \setminus X'$, with $X' \in \text{Conf}_s(\mathcal{E}')$.

Check that $X \cup X' \in \text{Conf}_s(\mathcal{E})$. Since $X \in \text{Conf}_s(\mathcal{E})$ (by definition of $X'$), $X$ (or $X'$) is finite, conflict-free, left-closed up to conflicts, does not contain flow cycles, and $\emptyset \rightarrow_{s} X$ in $\mathcal{E}$ (by $\mathcal{E}' \rightarrow_{s} X'$ in $\mathcal{E}'$).

First, it is clear that $X \cup X'$ is a finite subset of events of $\mathcal{E}$, by the definition of $E'$.

Second, $X \cup X'$ is a conflict-free set, due to Lemma 3(ii(b)).

Third, we verify that $X \cup X'$ is left-closed up to conflicts. Suppose that $e \in X \cup X'$, $d \prec e$ and $d \not\in X \cup X'$ for some $d \in X$. We have shown that there is $f \in X \cup X'$ such that $d \not\prec f \prec e$. If $e \in X$, then the result follows from the left-closedness up to conflicts of $X$. Consider the case when $e \notin X$, i.e. $e \in X' \subseteq E'$. Moreover, we have $d \in E'$ because $d \notin X$. If $\neg(d \not\prec e)$ then $(d,e) \notin \mathcal{E}'(X)$, i.e. $d \not\prec f \prec e$, for some $f \in X$. Check the case when $d \not\prec e$.

Since $d \notin X'$ and $X'$ is left-closed up to conflicts, there is $f \in X'$ such that $d \not\prec f \prec e$. Hence, $f \prec e$, and, moreover, $d \not\prec e$, because $d \not\prec f$.

Fourth, we verify that $X \cup X'$ does not contain flow cycles, i.e., $\leq_{(X \cup X')} := (\prec \cap ((X \cup X') \times (X \cup X')))^{*}$ is an antisymmetric relation. Assume that $x \leq_{(X \cup X')} x'$, i.e. $x = x_{0} \prec_{(X \cup X')} x_{1} \prec_{(X \cup X')} \ldots \prec_{(X \cup X')} x_{m} = x'$ ($m \geq 1$), and $x \neq x'$. We have to show that $\neg(x' \leq_{(X \cup X')} x)$. Consider possible cases.
• \( x' \in X \). Due to Lemma 3(ii(a(\(*\))))\), we have that \( x_i \in X \), for all \( 0 \leq i < m \). The result follows from the fact that \( X \) does not contain flow cycles.

• \( x' \in X' \).

* \( x \in X \). Suppose a contrary, i.e. \( x' \preceq_{(X \cup X')} x \). This means that \( x' = x'_0 \prec (X \cup X') \rightarrow x'_1 \prec (X \cup X') \rightarrow \cdots \rightarrow x'_{l-1} \prec (X \cup X') \rightarrow x'_l = x \) (\( l \geq 1 \)). Due to Lemma 3(ii(a(\(*\))))\), we get that \( x'_j \in X \), for all \( 0 \leq i < l \), contradicting \( x' \in X' \).

* \( x \in X' \). By Lemma 3(ii(a(\(*\))))\), we get that \( x_i \in X' \), for all \( 0 < i < m \). Due to Lemma 3(ii(a(\(*\))))\), \( x = x_0 \prec (X \cup X') \rightarrow x_1 \prec (X \cup X') \rightarrow \cdots \rightarrow x_m = x' (m \geq 1) \), i.e. \( x \preceq x' \). Clearly, \( x \not\preceq x' \). As \( X' \) does not contain flow cycles, it holds that \( \neg(x' \preceq x) \), i.e. \( \neg(x' = x_0 \prec (X \cup X') \rightarrow x_1 \prec (X \cup X') \rightarrow \cdots \rightarrow x_k = x) \) (\( k \geq 1 \)). Suppose a contrary, i.e. \( x' \preceq (X \cup X') x \). This implies that \( x' = x'_0 \prec (X \cup X') \rightarrow x'_1 \prec (X \cup X') \rightarrow \cdots \rightarrow x'_{k-1} \prec (X \cup X') \rightarrow x'_k = x \) (\( k \geq 1 \)), and, moreover, \( x'_{j} \in X \), for some \( 1 \leq j < k \), contradicting Lemma 3(ii(a(\(*\))))\).

Hence, \( X \cup X' \) is a configuration of \( E \). Clearly, \( \emptyset \rightarrow^\ast X \cup X' \). Thus, \( X \cup X' \) is a configuration in \( \ast \)-semantics of \( E \).

We shall prove that \( E' \setminus X' = E'' = \tilde{E} = E \setminus (X \cup X') \).

By definition, \( \tilde{E} = E \setminus (X \cup X') = (E \setminus X) \setminus (X \cup X') = E'' \). Due to Lemma 3(ii(b))\), \( \tilde{E}'' = (\tilde{E}' \cap (E'' \times E'')) \cup \{(a, a) \mid a \in \tilde{E}'(X')\} = ((\tilde{E}' \cap (E' \times E')) \cup \{(a, a) \mid a \in \tilde{E}'(X)\}) \cap (E'' \times E'') \cup \{(a, a) \mid a \in \tilde{E}'(X)\} = (\tilde{E}' \cap (E' \times E')) \cup \{(a, a) \mid a \in \tilde{E}'(X)\} = (\tilde{E}' \cap (E' \times E')) \cup \{(a, a) \mid a \in \tilde{E}(X \cup X')\} = \tilde{E}'' \).

Consider arbitrary \( a, b \in E'' \) such that \( a \preceq'' b \). This means that \( a, b \not\in X' \), \( a \not\prec b \), and \( (a, b) \not\in \tilde{E}'(X') \), i.e. \( \neg(a \prec b) \) or \( \neg(a \not\prec b) \), for all \( x' \in X' \). Then, \( a, b \not\in X, a \sim b, \) and \( (a, b) \not\in \tilde{E}'(X) \), i.e. \( \neg(x \prec b) \) or \( \neg(x \sim b) \), for all \( x \in X \). So, \( a, b \not\in X \cup X' \). We shall show that \( a \preceq b \), i.e. \( (a, b) \not\in \tilde{E}'(X \cup X') \).

Suppose a contrary, i.e. \( a \not\preceq b \), for some \( y \in (X \cup X') \). If \( y \in X \) then \( \neg(a \not\preceq b) \). Hence, \( y \in X' \). As \( a \not\in X \), we get that \( y \not\in a \). Moreover, \( y \not\preceq a \), by the definition of \( \tilde{E}' \). This implies that \( \neg(y \not\preceq b) \). As \( y \not\preceq b \), we have that \( (y, b) \not\in \tilde{E}' (X) \). Then, there is \( d \in X \) such that \( y \not\preceq d \prec b \). Hence, \( y \in \tilde{E}'(X) \).

Again by the definition of \( \tilde{E}' \), \( y \not\preceq y \), contradicting the conflict-freeness of \( X' \).

So, \( \neg(y \not\preceq b) \) or \( \neg(y \not\preceq a) \), for all \( y \in (X \cup X') \). Thus, \( a \sim b \).

Conversely, consider arbitrary \( a, b \in E'' \) such that \( a \preceq b \). Hence, \( a, b \not\in (X \cup X') \), i.e. \( a, b \in E'', a \not\prec b, \) and \( (a, b) \not\in \tilde{E}'(X \cup X') \), i.e. \( \neg(z \prec b) \) or \( \neg(z \sim b) \), for all \( z \in (X \cup X') \). This means that \( a \prec b \). We shall show that \( a \prec'' b \), i.e. \( (a, b) \not\in \tilde{E}'(X) \). Suppose a contrary, i.e. \( a \not\prec b \), for some \( x' \in X' \).

Then, \( a \in \tilde{E}'(X') \). As \( a \in E'' \), this implies that \( a \prec'' a \), by the definition of \( \tilde{E}'' \). Since \( x' \in X' \) and \( a \not\in X' \), we get that \( x' \not\preceq a \). Moreover, \( x' \not\preceq a \), by virtue of the definition of \( \tilde{E}' \). Due to \( x' \in (X \cup X') \), we get \( \neg(x' \prec b) \), contradicting \( x' \prec b \). Thus, \( a \prec'' b \).

Finally, we get that \( l'' = \left. l \right|_{E''} = \left. l \right|_{\tilde{E}} = \tilde{l} \).

(ii) Let \( X \rightarrow X'' \) in \( E \). Since \( X \) and \( X'' \) are configurations of \( E \), \( X \) and \( X'' \) are finite subsets of \( E \), conflict-free, left-closed up to conflicts and do not contain flow cycles.
First, it is clear that $X' = X'' \setminus X$ is a finite subset of $E'$, by the definition of $E'$.

Second, we show that $X'$ is a conflict-free subset of $E'$. Suppose a contrary, i.e. there is $e, e' \in X'$ such that $e * e'$. Obviously, $X'$ is a conflict-free subset of $E$. Then, $\neg(e * e')$. Due to the definition of $\neg$, $e = e'$ and $e \in E(X)$. Hence, there is $e'' \in X$ such that $e * e''$, contradicting the conflict-freeness of $X''$.

Third, we check that $X'$ is left-closed up to conflicts. Assume that $e \in X'$, $d \prec e$, and $d \not\in X'$, for some $d \in E'$. We have to show that there is $f \in X'$ such that $d * f \prec e$. By virtue of the definition of $\prec$, $d * e$ and (**) $(d, c) \not\prec e(X)$, i.e. $\neg(x * e)$ or $\neg(x * d)$, for all $x \in X$. Due to (**), $e \in E''$, and $X''$ being left-closed up to conflicts, either case is admissible.

* $d \in X''$. As $d \not\in X'$, we get $d \in X$, contradicting $d \not\in E'$.

* $\exists f \in X''$ s.t. $d * f \prec e$. Due to (**), $f \not\in X$. Hence, $f \in X'$. As $d \not\in X'$, we get $f \not\neq d$, and, moreover, $f * d$, because $f, d \in E'$ and $f * d$. Suppose a contrary, i.e. $\neg(f * e)$. Since $f * e$, by the definition of $\prec$, we have that $(f, e) \not\prec e(X)$, i.e. there is $z \in X$ such that $f * z \prec e$. As $X, X' \subseteq X''$, $z \in X$, and $f \in X'$, this contradicts the conflict-freeness of $X''$. So, $f * e$.

Fourth, we verify that $X'$ does not contain flow cycles, i.e., $\leq_{X'} := (\prec_{X'} \cap (X' \times X'))^*$ is an antisymmetric relation. Assume that $x \leq_{X'} x'$, i.e. $x = x_0 \prec_{X'} x_1 \ldots \prec_{X'} x_n = x'$ $(m \geq 1)$, and $x \neq x''$, for some $x, x' \in X'$. By the definition of $\prec$, we have that $x = x_0 \prec_{X'\bar{x}} x_1 \ldots \prec_{X'\bar{x}} x_m = x'$ $(m \geq 1)$. Suppose that there is $0 \leq i \leq m$ and $\bar{x} \in X$ such that $x_i \prec_{X'\bar{x}} \bar{x}$. Since $X = X'' \setminus X', x_1 \not\in X$. Due to $X$ being left-closed up to conflict, we can find $z \in X$ such that $x_i \prec_{X'\bar{x}} z$, $x_i \prec_{X'\bar{x}} \bar{x}$, contradicting the conflict-freeness of $X''$. Then, we get that $\neg(x_i \prec_{X'\bar{x}} \bar{x})$, for all $i \leq m$ and $\bar{x} \in X$. Hence, due to $X'' = X \cup X'$, it holds that $x = x_0 \prec_{X'\bar{x}} x_1 \ldots \prec_{X'\bar{x}} x_m = x'$ $(m \geq 1)$, i.e. $x \leq_{X''} x'$ and $x \neq x''$. As $X''$ does not contain flow cycles, we have that $\neg(x' \leq_{X''} x)$, i.e. $\neg(x' = x_0 \prec_{X'\bar{x}} x_1 \ldots \prec_{X'\bar{x}} x_l = x)$ $(l \geq 1)$. So, it is true that $\neg(x' = x_0 \prec_{X'\bar{x}} x_1 \ldots \prec_{X'\bar{x}} x_l = x)$ $(l \geq 1)$, because $X' \subseteq X''$. Thus, $\neg(x' \leq_{X''} x)$.

Clearly, $\emptyset \not\rightarrow X' \in E' \setminus X$. Thus, $X'$ is a configuration in $*$-semantics of $E' \setminus X$. □

**Case with** $E \in E^{\not\rightarrow}_L$ and * = mset.

**Proof.** (i) Let $E' = E \setminus X$ with $X \in \text{Conf}_*(E)$ and $E'' = E' \setminus X'$ with $X' \in \text{Conf}_*(E')$.

Since $X \in \text{Conf}_*(E)$ ($X' \in \text{Conf}_*(E')$), $X$ ($X'$) is a finite, conflict-free, and secured subset of $E$ ($E'$) and $\emptyset \rightarrow X \in E$ ($\emptyset \rightarrow X' \in E'$). Obviously, $X \cup X'$ is a finite subset of $E$, by the definition of $E'$. Next, due to Lemma 4(iii), we have that $X \cup X' \in \text{Con}$, i.e. it is conflict-free. Further, we check that $X \cup X'$ is secured. As $X$ is secured, there is a sequence $e_1 \ldots e_n$ $(n \geq 0)$ such that $X = \{e_1, \ldots, e_n\}$, and $\{e_1, \ldots, e_i\} \vdash e_{i+1}$, for all $i < n$. Moreover, there exists a sequence $e'_1 \ldots e'_m$ $(m \geq 0)$ such that $X' = \{e'_1, \ldots, e'_m\}$, and $\{e'_1, \ldots, e'_j\} \vdash e'_{j+1}$, for all $j < m$, because $X'$ is secured in $E'$. Consider a sequence $e_1 \ldots e_n e_{n+1} =
\[ \eta_1 \ldots \eta_{n+m} = \eta_m. \] Clearly, \( \{ \eta_1, \ldots, \eta_n, \eta_{n+1}, \ldots, \eta_{n+m} \} = X \cup X'. \) Check that \( \{ \eta_1, \ldots, \eta_n \} \vdash \eta_{n+1}, \) for all \( l < n + m. \) We know that \( \{ \eta_1, \ldots, \eta_l \} \vdash \eta_{l+1}, \) for all \( l < n. \) Consider the case when \( l = n. \) Obviously, \( \emptyset \vdash \eta_n, i.e. \emptyset \vdash \eta_{n+1} = \eta_1. \) This means that there is \( (W, \eta_1) \in \mathcal{F}_{\min} \) such that \( \emptyset = W \cap E \) and \( \{ \eta_1 \} \cup X \in \mathcal{C}. \) Since \( W \subseteq E' \cup X \) and \( W \cap E' = \emptyset, \) we get \( \emptyset \subseteq X \in \mathcal{C}. \)

Then, \( X \vdash \eta_{n+1}, i.e. \{ \eta_1, \ldots, \eta_n \} \vdash \eta_{n+1}. \) Finally, we verify the case when \( n+1 \leq l < n+m. \) We know that \( \{ \eta_1, \ldots, \eta_l \} \vdash \eta_{l+1}, \) for some \( W' \subseteq \{ \eta_1, \ldots, \eta_l \} \). This means that there is \( (W, \eta_{l+1}) \in \mathcal{F}_{\min} \) such that \( W' = W \cap E', \{ \eta_1, \ldots, \eta_l \} \cup X \in \mathcal{C}, \) and \( W' \cap X \in \mathcal{C}. \) So, \( W' \cup X \vdash \eta_{l+1}. \) Since \( X \cup X' \in \mathcal{C} \) and \( W' \subseteq \{ \eta_1, \ldots, \eta_n \} \subseteq X', X \cup \{ \eta_1, \ldots, \eta_n \} \in \mathcal{C}. \) Then, \( \{ \eta_1, \ldots, \eta_n \} \vdash \eta_{n+1}. \) Hence, \( X \cup X' \) is a configuration of \( \mathcal{E}. \) Since \( \emptyset \subseteq X \cup X' \) we get \( \emptyset \rightarrow X \cup X' \in \mathcal{C}. \) Thus, \( X \cup X' \in \mathcal{F}_{\max}(\mathcal{E}). \)

Set \( \tilde{E} = \mathcal{E} \setminus (X \cup X'). \) Check that \( \mathcal{E}' = \tilde{E}. \) By definition, \( \tilde{E} = E' \setminus (X \cup X') = (E \setminus X) \setminus X' = E' \setminus X' = E'. \) Then, again by definition, \( \mathcal{E}' = (E \setminus X) \setminus X' = (E \setminus X) \setminus (X' \setminus E') = (E \setminus X) \setminus (E' \setminus X). \) By definition \( \mathcal{E}' = (E \setminus X) \setminus (E' \setminus X), \) and \( \mathcal{E}' = (E \setminus X) \setminus (E' \setminus X) \).

Using that \( \tilde{E} = E' = \mathcal{E}' \) and \( \tilde{E} = \mathcal{E}' \), it is straightforward to verify that \( \mathcal{C} = \mathcal{C}'. \) It remains to show that \( \emptyset = \mathcal{C}'. \) We know that \( \mathcal{F}_{\min} = \{ (W, \eta) \mid \exists (W, \eta) \in \mathcal{F}_{\min} \text{ s.t. } e \in E, W = W' \cap E, \{ e \} \cup (X \cup X') \in \mathcal{C}, \text{ and } W' \cup (X \cup X') \subseteq \mathcal{C} \}. \)

On the other hand, we have that \( \mathcal{F}_{\min} = \{ (W', \eta) \mid \exists (W', \eta) \in \mathcal{F}_{\min} \text{ s.t. } e \in E', W' = W' \cap E', \{ e \} \cup X \subseteq \mathcal{C}, \{ e \} \cup X' \subseteq \mathcal{C}, \text{ and } W' \cup X \subseteq \mathcal{C} \}. \) As \( \tilde{W} = \mathcal{W}(X \cup X'), \tilde{W} \cup X = X \cup X', \) and as \( \tilde{W} = W' \setminus X, \tilde{W} \cup X = W' \cup X = (W \setminus X) \cup X'. \) Hence, \( \tilde{W} \cup X \subseteq \mathcal{C} \) if \( \tilde{W} \cup X' \subseteq \mathcal{C} \) and \( \tilde{W} \cup X' \subseteq \mathcal{C}, \) due to Lemma 4(iii). Notice that \( e \notin X \cup X', \) because \( e \in E'. \) So, \( \{ e \} \cup X \subseteq \mathcal{C} \) and \( \{ e \} \cup X' \subseteq \mathcal{C}, \) again by Lemma 4(iii). Due to \( \tilde{E} = E', \) we get that \( \mathcal{F}_{\min} = \mathcal{F}_{\min} \). Moreover, it holds that \( \tilde{E} = \{ (W, e) \mid W \in \mathcal{C}, \exists (W, e) \in \mathcal{F}_{\min} \text{ s.t. } W \subseteq \tilde{W} \} = \{ (W', e) \mid W \in \mathcal{C}, \exists (W, e) \in \mathcal{F}_{\min} \text{ s.t. } W \subseteq W'. \} \subseteq \mathcal{C} = \mathcal{C}'. \) As \( \tilde{W} \subseteq W'. \)

(ii) Assume \( X, X' \subseteq \mathcal{F}_{\max}(\mathcal{E}) \) and \( X \rightarrow X'. \) Since \( X' \subseteq \mathcal{F}_{\max}(\mathcal{E}), i.e. X' \in \mathcal{C}, \) and \( \mathcal{C} \) is finite, conflict-free (i.e., \( X' \subseteq \mathcal{C} \)), and \( \mathcal{C} \) is secured (i.e., there exists a sequence \( t = \eta_1 \ldots \eta_n \) of events from \( E \cup \emptyset \)) such that \( X' = \{ \eta_1, \ldots, \eta_k \}, \) and \( \{ \eta_1, \ldots, \eta_k \} \vdash \eta_{k+1}, \) for all \( i < n. \) Using the sequence \( t \) and the fact that \( X' \subseteq \mathcal{C}, \) we construct a sequence \( t' \) as follows: \( t'_0 = \epsilon, t'_{i+1} = t_i \eta_{i+1}, \) if \( \eta_{i+1} \notin X, \) and \( t'_{i+1} = t_i, \) otherwise. W.l.o.g. assume \( t' = \eta_1 \ldots \eta_k, \) for \( k \geq 0 \) and \( X' = \{ \eta_1, \ldots, \eta_k \}. \) Clearly, \( X' \) is a finite subset of \( E'. \) As \( X'_n \subseteq \mathcal{C}, \) then \( X' = X' \cap E' \subseteq \mathcal{C} \), by Lemma 4(i). We shall now show that \( \{ \eta_1, \ldots, \eta_k \} \vdash \eta_{k+1}, \) for all \( j < k. \) Take an arbitrary \( j < k. \) Clearly, \( \eta_1, \ldots, \eta_j, \eta_{j+1} \in X' \subseteq E \) and \( \{ \eta_1, \ldots, \eta_j \} \subseteq \mathcal{C}. \) W.l.o.g. assume \( \eta_{j+1} = \eta_{j+1}, \) for some \( \eta_{j+1} \in X'. \) As \( X' \subseteq \mathcal{C}, \) we get that \( \mathcal{E}' \subseteq \{ \eta_1, \ldots, \eta_j \} \vdash \eta_{j+1}. \) Then, \( \emptyset \vdash \mathcal{E}' \cup X \subseteq \mathcal{C}. \) W.l.o.g. \( \mathcal{E}' \cup X \subseteq \mathcal{C}. \) Set \( W' = W \cap E'. \) Clearly, \( W' \cup \{ \eta_{i+1} \} \cup X \subseteq X'. \) As
\(X'' \in \text{Con},\) we get that \(W' \cup X \in \text{Con} \) and \(\{e_{j+1}''\} \cup X \in \text{Con}.)\) Therefore, \(W' \vdash_{\text{min}} e_{j+1}''.\)

Since \(W' \subseteq \{e_1', \ldots, e_j'\} \in \text{Con', \} \{e_1, \ldots, e_j'\} \vdash e_{j+1}'.\) This implies, \(X'\) is a configuration of \(E \setminus X'.\) Since \(\emptyset \subseteq X',\) we have \(\emptyset \rightarrow_{\ast} X' = X'' \setminus X \in E \setminus X'.\)

Hence, \(X' \in \text{Conf}_{\ast}(E \setminus X).\)

\(\square\)

**Case with \(E \in \mathbb{E}_{L}^i\) and \(\ast = \text{step}.\)**

**Proof.** (i) Let \(E' = E \setminus X,\) with \(X \in \text{Conf}_{\ast}(E),\) and \(E'' = E' \setminus X',\) with \(X' \in \text{Conf}_{\ast}(E').\)

Check that \(X \cup X' \in \text{Conf}_{\ast}(E).\) Since \(X \in \text{Conf}_{\ast}(E) (X' \in \text{Conf}_{\ast}(E'))\), \(X \cup X') is a finite subset of \(E \cup E',\) and \(\emptyset \rightarrow_{\ast} E \) in \(E (\emptyset \rightarrow_{\ast} X' \in E').\)

This implies that \(\emptyset \rightarrow_{\ast} X_1 \ldots X_{n-1} \rightarrow_{\ast} X_n = X \in E \) for some \(n \geq 0\) \((\emptyset \rightarrow_{\ast} X_1 \ldots X_{m-1} \rightarrow_{\ast} X_m = X' \in E \) for some \(m \geq 0).\) By the definition of \(C',\) we may conclude that \(X \cup X_1 \ldots X_{m-1} \subseteq C.\) It remains to show that \(\emptyset \rightarrow_{\ast} X_1 \ldots X_{n-1} \rightarrow_{\ast} X_n = X \rightarrow (X \cup X') \rightarrow (X \cup X_m) \rightarrow_{\ast} (X \cup X_m') \subseteq E.\) We shall proceed by induction on \(m.\)

\(m = 0.\) Obvious.

\(m > 0.\) By the induction hypothesis, \(\emptyset \rightarrow_{\ast} X_1 \ldots X_{n-1} \rightarrow_{\ast} X_n = X \rightarrow_{\ast} (X \cup X_1) \ldots (X \cup X_{m-2}) \rightarrow_{\ast} (X \cup X_{m-1}) \subseteq E.\) We have to show that \((X \cup X_{m-1}) \rightarrow_{\ast} (X \cup X_m) \in E.\) Since \(X_{m-1} \rightarrow_{\ast} X_m \subseteq E,\) we have \(X \cup X_{m-1} \subseteq X \cup X_m \) and for all \(Z \subseteq X \cup X_m \) if \(X_{m-1} \subseteq Z\) then \(Z \subseteq E.\) Clearly, \(X \cup X_{m-1} \subseteq X \cup X_m.\) Take an arbitrary set \(W \subseteq X \cup X_m \) such that \(X \cup X_m \subseteq W.\) This means that \(W = X \cup X'_m \) for some \(W' \subseteq X_m.\) Obviously, \(X' \subseteq X'.\) Hence, \(W' \subseteq E.\) This implies that \(X \cup W' = W \in C,\) by the definition of \(C'.\) So, \(X \cup W' \rightarrow_{\ast} (X \cup X_m) \subseteq E.\)

Thus, \(X \cup X' \in \text{Conf}_{\ast}(E).\)

Check that \(E'' = E \setminus (X \cup X') = \bar{E}.\) Clearly, \(E'' = E' \setminus X' = (E \setminus X) \setminus X' = E \setminus (X \cup X') = \bar{E}.\) Moreover, it holds that \(C'' = \{X \subseteq E'' \mid X \cup X' \cup Y \subseteq C\} = \{Y \subseteq E'' \mid X \cup X' \cup Y \subseteq C\} = \bar{C}.\) Obviously, \(i'' = i'.\)

(ii) \((\Rightarrow)\) Let \(X \rightarrow_{\ast} X'' \) in \(E.\) This implies that \(X \subseteq X''\) and for all \(Z \subseteq X''\) if \(X \subseteq Z\) then \(Z \subseteq E.\) Let \(X' = X'' \setminus X.\) As \(X \cup X' \subseteq C,\) we get \(X' \subseteq C',\) by the definition of \(C'.\) Take an arbitrary set \(\emptyset \subseteq W \subseteq X'.\) Then, \(X \subseteq W \subseteq X \cup X' \subseteq X'' = X''.\) Hence, \(W \subseteq E.\) This implies that \(W \in C',\) again by the definition of \(C'.\) So, \(\emptyset \rightarrow_{\ast} X' \in E \setminus X\) and \(X' \in \text{Conf}_{\ast}(E \setminus X).\)

\(\square\)

**Case with \(E \in \mathbb{E}_{L}^i\) and \(\ast = \text{step}.\)**

**Proof.** (i) Let \(E' = E \setminus X\) with \(X \in \text{Conf}_{\ast}(E),\) and \(E'' = E' \setminus X'\) with \(X' \in \text{Conf}_{\ast}(E').\)

Check that \(X \cup X' \in \text{Conf}_{\ast}(E).\) Since \(X \in \text{Conf}_{\ast}(E),\) \(X \in \text{LC}(E)\) and \(\emptyset \rightarrow_{\ast} X_1 \ldots X_{n-1} \rightarrow_{\ast} X_n = X \) \((n \geq 0)\) in \(E.\) As \(X' \in \text{Conf}_{\ast}(E'),\) \(X' \in \text{LC}(E')\) and \(\emptyset \rightarrow_{\ast} X_1 \ldots X_{m-1} \rightarrow_{\ast} X_m = X' \) \((m \geq 0)\) in \(E'.\) By Lemma 7(ii), \(X \cup X' \in \text{LC}(E).\) It remains to show that \(\emptyset \rightarrow_{\ast} X_1 \ldots X_{n-1} \rightarrow_{\ast} X_n = X \rightarrow_{\ast} (X \cup X') \rightarrow_{\ast} (X \cup X_m) \rightarrow_{\ast} (X \cup X_m') \subseteq E.\) We shall proceed by induction on \(m.\)
m = 0. Obvious.

m > 0. By the induction hypothesis, \( \emptyset \to_* X_1 \ldots X_{n-1} \to_* X_n = X \to_* (X \cup X'_1) \ldots (X \cup X'_{m-2}) \to_* (X \cup X'_{m-1}) \) in \( \mathcal{E} \). We have to show that \((X \cup X'_{m-1}) \to_* (X \cup X'_m)\) in \( \mathcal{E} \). By Proposition 3.11 from [13], we can w.l.o.g. assume that \( |X'_m \setminus X'_{m-1}| = 1 \). Then, we have that \( (X \cup X'_{m-1}) \to_* (X \cup X'_m) \) in \( \mathcal{E} \).

Thus, \( X \cup X' \in \text{Conf}_*(\mathcal{E}) \).

Clearly, \( E'' = E \setminus X \setminus X' = E \setminus (X \cup X') = \bar{E} \). Moreover, we have that \( \bar{F} = \{(A, B) \mid \exists (A, B) \in \bar{F} \text{ s.t. } A = A \cap E \setminus (X \cup X'), B = B \cap E \setminus (X \cup X')\} \). On the other side, we get that \( \bar{t} = \{(A', B') \mid \exists (A', B') \in \bar{t} \text{ s.t. } A' = A' \cap (E \setminus X \setminus X'), B' = B' \cap (E \setminus X \setminus X'), (A'' \cup B'' \cup X'') \in LC(E')\} = \{(A', B') \mid \exists (A' \cup B' \cup X') \in LC(E\setminus E')\} \) that \( \bar{t} = \bar{t} \). Obviously, \( \bar{t}' = \bar{t}' \).

(ii) Let \( X \to_* X'' \) in \( \mathcal{E} \). Then, \( X \subseteq X'' \) and \( X' \subseteq X' \). Due to Lemma 7(ii), we get that \( \bar{X} \subseteq \bar{X} \subseteq \bar{X} \subseteq \bar{X} \subseteq \bar{X} \).

Proposition 2. Given a structure \( \mathcal{E} \) over \( L \),

(i) for any \( X \in \text{Conf}_*(\mathcal{E}) \), \( E \setminus X \in \text{Reach}_*(\mathcal{E}) \);
(ii) for any \( \mathcal{E}' \in \text{Reach}_*(\mathcal{E}) \), there exists \( X \in \text{Conf}_*(\mathcal{E}) \) such that \( \mathcal{E}' = E \setminus X \);
(iii) for any \( X', X'' \in \text{Conf}_*(\mathcal{E}) \), if \( \mathcal{E}' \models X' \), then \( \mathcal{E} \setminus X' \models \mathcal{E} \setminus X'' \);
(iv) for any \( \mathcal{E}' \in \text{Reach}_*(\mathcal{E}) \), if \( \mathcal{E}' \models X'' \), then there are \( X', X'' \in \text{Conf}_*(\mathcal{E}) \) such that \( \mathcal{E}' = E \setminus X' \), \( \mathcal{E}'' = E \setminus X'' \), \( X' \models \mathcal{E}'' \).

Proof. Consider the cases with \( * \in \{\text{int, step, mset, pom}\} \).

(i) Take an arbitrary set \( X \in \text{Conf}_*(\mathcal{E}) \). This means that \( X_0 = \emptyset \to_* X_1 \ldots X_{n-1} \to_* X_n = X \) in \( \mathcal{E} \). This implies that \( X_0, X_1, \ldots, X_n \in \text{Conf}_*(\mathcal{E}) \) and \( X_{i-1} \subseteq X_i \) for all \( i < n \). We shall proceed by induction on \( n \).

\( n = 0 \). \( E \setminus X_0 = E \setminus \emptyset = E \in \text{Reach}_*(\mathcal{E}) \).

\( n = 1 \). By the induction hypothesis, \( E \setminus X_0 \in \text{Reach}_*(\mathcal{E}) \). Check that \( E \setminus X_0 \models p_1 \).

\( E \setminus X_1 \). Since \( X_0 \to_* X_1 \) in \( \mathcal{E} \), it holds \( X_1 \setminus X_0 \in \text{Conf}_*(\mathcal{E} \setminus X_0) \) and \( \emptyset \to_* X_1 \setminus X_0 = A_1 \) in \( \mathcal{E} \setminus X_0 \), by Proposition 1(ii). Next, by Proposition 1(i), we have \( (E \setminus X_0) \setminus X_0 = E \setminus (X_0 \cup (X_1 \setminus X_0)) = E \setminus X_1 \). This implies that \( E \setminus X_0 \models X_1 \), where \( p_1 = l_*(A_1) \). So, \( E \setminus X_1 \in \text{Reach}_*(\mathcal{E}) \).

\( n > 1 \). By the induction hypothesis, \( E \setminus X_{n-2} \models p_{n-1} \).

Reasoning as in the previous item, we get that \( E \setminus X_{n-2} \models X_{n-1} \), where \( p_n = l_*(A_n) \). Thus, \( E \setminus (X_n = X) \in \text{Reach}_*(\mathcal{E}) \).
(ii) Take an arbitrary $E' \in \text{Reach}_s(E)$. This means that $E = E_0 \xrightarrow{p_1} E_1 \ldots \xrightarrow{p_{n-1}} E_{n-1} \xrightarrow{p_n} E_n = E'$ $(n \geq 0)$. By the definition of $p_{n+1}$, it holds that $E_{i+1} = E_i \setminus X_{i+1}$, for some $X_{i+1} \in \text{Conf}_s(E_i)$ such that $\emptyset \rightarrow_{s} X_{i+1}$ in $E_i$ and $p_{i+1} = l_s(X_{i+1})$ $(i < n)$. Verify that $Y_{i+1} = \bigcup_{j=1}^{i+1} X_j \in \text{Conf}_s(E)$ and $E_{i+1} = E \setminus Y_{i+1}$, for all $i < n$. We shall proceed by induction on $n$.

$n = 0$. Obvious.

$n = 1$. Then, $Y_1 = X_1 \in \text{Conf}_s(E)$ and $E_1 = E_0 \setminus X_1 = E \setminus Y_1$.

$n > 1$. By the induction hypothesis, $Y_{n-1} = \bigcup_{j=1}^{n-1} X_j \in \text{Conf}_s(E)$ and $E_{n-1} = E \setminus Y_{n-1}$. Check that $Y_n = \bigcup_{j=1}^{n} X_j \in \text{Conf}_s(E)$ and $E_n = E \setminus Y_n$. As $E_n = E_{n-1} \setminus X_n$, it holds that $E_n = (E \setminus Y_{n-1}) \setminus X_n$. According to Proposition 1(i), we have that $Y_{n-1} \cup X_n \in \text{Conf}_s(E)$ and $E_n = E \setminus (Y_{n-1} \cup X_n) = E \setminus Y_n$. Thus, $E' = E \setminus Y_n$.

(iii) It follows from the definitions of the transition relations and items (i) and (ii) of Proposition 1.

(iv) Due to item (ii), there is $X' \in \text{Conf}_s(E)$ such that $E' = E \setminus X'$. According to the definition of $p_{s}$, we have that there is $\tilde{X}' \in \text{Conf}_s(E')$ such that $E'' = E' \setminus \tilde{X}'$, $\emptyset \rightarrow_{s} \tilde{X}'$ in $E'$ and $p = l_s(\tilde{X}')$. Then, $X'' = X' \cup \tilde{X}' \in \text{Conf}_s(E)$ and $E'' = E \setminus X''$, by item (i) of Proposition 1. Consider three possible cases:

$\star = \text{int}$ Since $\emptyset \rightarrow_{\text{int}} \tilde{X}'$ in $E'$, we have $\tilde{X}' = \{e\}$. Hence, $X'' \setminus X' = \tilde{X}' = \{e\}$ and $X' \subseteq X''$. Thus, $X' \rightarrow_{\text{int}} X''$ in $E$.

$\star = \text{step}$ Since $\emptyset \rightarrow_{\text{step}} X'$ in $E'$, we get, for all $\emptyset \subseteq A \subseteq \tilde{X}'$, $A \in \text{Conf}(E')$.

By definition of $\rightarrow_{\text{step}}$, we have $\emptyset \rightarrow_{\text{step}} A$ in $E'$, for all $\emptyset \subseteq A \subseteq \tilde{X}'$. Hence, $A \in \text{Conf}_{\text{step}}(E')$. Then, by item (i) of Proposition 1, $X''' = X' \cup A \in \text{Conf}_{\text{step}}(E)$, for all $\emptyset \subseteq A \subseteq \tilde{X}'$. It is clear that for all $X' \subseteq X \subseteq X'' = X' \cup \tilde{X}'$, there is $A \subseteq \tilde{X}'$ such that $X = X' \cup A$. Hence, $X' \rightarrow_{\text{step}} X'''$ in $E$.

$\star = \text{mset}$ It is clear that $X' \subseteq X' \cup \tilde{X}' = X''$. Hence, $X' \rightarrow_{\text{mset}} X''$ in $E$.

$\star = \text{pom}$ Since $\emptyset \rightarrow_{\text{pom}} \tilde{X}'$ in $E'$, we have $\leq_{\tilde{X}'}$ is defined. It is clear that $X' \subseteq X' \cup \tilde{X}' = X''$. Moreover, $X'' \setminus X' = X'$. Hence, $\leq_{X'' \setminus X'} = \leq_{\tilde{X}'}$ is defined. So, $X' \rightarrow_{\text{pom}} X''$ in $E$.

Obviously, $l_s(X'' \setminus X') = p$. Thus, $X' \rightarrow_{s} X''$ in $E$. □

**Theorem 1.** Given a structure $E$ over $L$, $T_{C_1}(E)$ and $TR_s(E)$ are isomorphic.

**Proof.** Define a mapping $g : \text{Conf}_s(E) \rightarrow \text{Reach}_s(E)$ as follows: $g(X) = E \setminus X$, for all $X \in \text{Conf}_s(E)$. Clearly, $g(\emptyset) = E$. By the definition of $E \setminus X$ and Proposition 2 (i), $g$ is well-defined. Check that $g$ is a bijective mapping.

Suppose that $g(X) = g(X')$, for some $X, X' \in \text{Conf}_s(E)$. This means that $E \setminus X = E \setminus X'$. By the definition of $E \setminus X$ and $E \setminus X'$, we get that $E \setminus X = E \setminus X'$. Since $X, X' \subseteq E$, we have that $X = X'$. Thus, $g$ is an injective mapping.

Take an arbitrary $E' \in \text{Reach}_s(E)$. Due to Proposition 2 (ii), we get that $E' = E \setminus X$, for some $X \in \text{Conf}_s(E)$. So, $g$ is a surjective mapping.
We finally show that $X \xrightarrow{P} X'$ in $TC_\epsilon(\mathcal{E})$ if $g(X) \xrightarrow{P} g(X')$ in $TR_\epsilon(\mathcal{E})$. It follows from items (iii) and (iv) of Proposition 2 and the fact that $g$ is a bijective mapping.

Thus, $g$ is indeed an isomorphism. $\Box$